The Algorithmics Column

by

Gerhard J Woeginger

Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven, The Netherlands
gwoegi@win.tue.nl

Computational Aspects of Packing Problems

Helmut Alt *

Abstract

Packing problems have been investigated in mathematics since centuries. For polygons, circles, or other objects bounded by algebraic curves or surfaces it can be argued that packing problems are computable. However, even some of the simplest versions in the plane turn out to be NP-hard unless the number of objects to be packed is bounded. This article is a survey on results achieved about computability and complexity of packing problems, about approximation algorithms, and about very natural packing problems whose computational complexity is unknown.

Packing objects is a quite natural problem and has been investigated in mathematics and operations research for a long time. Applications concern the physical nonoverlapping packing of concrete objects during storage or transportation but also in two dimensions how to efficiently cut prescribed pieces from cloth or sheet metal while minimizing waste. Even more abstract problems like, e.g., efficient scheduling with respect to time and space can be modelled as packing rectangles into a strip.

1 Variants of the problem

Formally, packing means that a set of mathematically modelled geometric objects is moved nonoverlapping into a mathematically modelled container. Numerous variants are possible depending on the following issues:

- *Objects* to be packed, e.g., rectangles, disks, convex polygons, nonconvex polygons, cuboids, polyhedra, spheres.

- *motions allowed* to be applied to the objects: translations or rigid motions (translation and rotation).

*Institute of Computer Science, Freie Universität Berlin, Takustr. 9, 14195 Berlin, Germany. alt@mi.fu-berlin.de .

- Packing into a *given container* C, i.e., the decision problem whether all given objects can be packed into C.

- There is an allowable set of containers, e.g., all rectangles or all convex sets. Find a *container of minimum size* (e.g., area, volume, perimeter) into which all objects can be packed.

- *Strip packing:* In two dimensions an infinite strip of a fixed width is given, pack the objects so that a minimum length of the strip is used. In three dimensions a strip is given by a rectangular cross section.

- *Bin packing* (an important application is the *cutting stock problem*): all containers have the same shape and size. Minimize the number of containers to pack all objects.

- *Online-* or *Offline-packing*: In offline-packing all objects to be packed are known in advance, in online-packing they arrive one-by-one and have to be packed upon arrival.

In this survey, we will concentrate on issues of offline-packing and not go into the details of the vast research that has been done on bin packing but rather discuss computability, computational complexity, describe some strip packing algorithms, and concentrate on approximation algorithms for finding minimum size containers for a set of given objects.

2 Classical Packing Problems

Packing is a vast field in mathematics, in particular in discrete geometry. For a survey on finite packings see, e.g., [6] or [14].

Probably, the most prominent packing problem in mathematics which was open for four centuries used to be known as *Kepler's conjecture*. The astronomer Johannes Kepler conjectured in 1611 that the most efficient way to pack infinitely many spheres in three dimensions is the way fruit sellers do it with oranges at the market: arrange the spheres in layers where within each layer the spheres form a hexagonal grid, see Figure 1. This way a density of $\pi/\sqrt{18} \approx .74$ is obtained. Gauss showed in 1831 that this arrangement is optimal among regular lattices. László Fejes Tóth showed in 1953 that the general proof could be reduced to a finite yet lengthy calculation. Finally, in an effort starting in the early 1990s and lasting for about 20 years a group of researchers around Thomas Hales managed to simplify the calculation and implement and run a computer program which was supposed to finalize the proof. After the proof has been completely formalized and

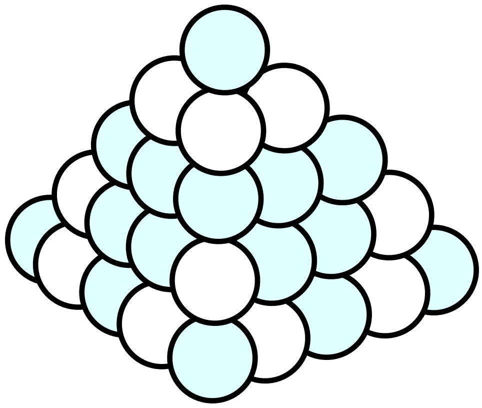

Figure 1: Packing spheres

verified by proof checking systems [15] most mathematicians agree that Kepler's conjecture has been proven.

For the corresponding problem in two dimensions, namely how to pack disks of equal radius so that the density is maximized it seems quite intuitive to pack them as a hexagonal grid. This fact is called Thue's Theorem but had been shown for lattices already by Lagrange in 1773 and a complete proof is due to Fejes Tóth.

Finding a minimum area container for packing finitely many unit disks is a variant of the problem that has gained considerable attention. Each number of disks to be placed into a minimum area circle, square, etc. constitutes a mathematical problem on its own. Webpages "Erich's packing center" by Erich Friedman http://www2.stetson.edu/~efriedma/packing.html and "Packomania" http://www.packomania.com/ show the best known solutions for various shapes of objects and containers, e.g., minimum area circular containers for k unit disks for $k = 1, ..., 2600$. Unfortunately, already $k = 12$ is the smallest number for which the best known solution has not yet been proven optimal, see Figure 2.

For unit spheres in three and higher dimensions, interesting properties and open problems about the so-called **sausage packing** have been found. Sausage packing means to place the spheres contiguously so that their centers lie on a straight line ("sausage"). The question is whether the sausage packing is the most efficient way of packing finitely many unit spheres, i.e., yields a minimum volume convex hull. Whereas in two dimensions this statement is obviously false for 3 or more disks, in three dimensions it can only shown to be true for at most four and false for 56 or more spheres.

For dimension five or higher, there is the *sausage conjecture* due to Fejes Tóth [11] that for any number of spheres the sausage packing is optimal. For dimensions greater than or equal 42 the sausage conjecture has been solved by Betke et al.

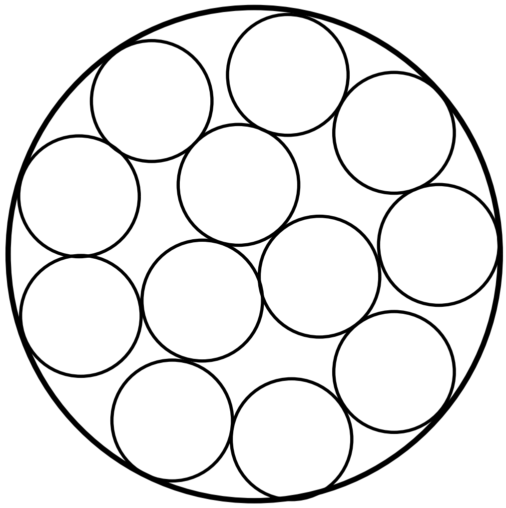

Figure 2: Packing 12 circles into a (possibly) minimum size circle

(1994) [5] and Betke and Henk [4].

In the case of packing k unit squares into minimum area disks for some values of k it is not optimal that all squares have the same orientation, i.e., packing under rigid motions (translations and rotations) gives a smaller solution than packing by translations only.

3 Computability

All packing problems under rigid motions or translations involving a container and a finite number of objects which can be described by algebraic equations or inequalties are *decidable*, or, in case of optimization problems, *computable*. This includes nearly all problems mentioned in the previous sections. The problems in connection with the sausage packings are an exception, however, since the containers are convex hulls of spheres and cannot easily be described.

In fact, in case of the decision problem assume that we have a set of objects and a container that are described by algebraic formulas. The question is whether the objects can be fit into the container by certain motions (translations or rigid motions). Let $F(x)$ be the formula describing the container in the sense that a point with coordinate vector x is in the container exactly if $F(x)$ is true. In the same way, let $F_i(t_i, x)$ describe the i-th object, $i = 1, ..., n$, where t_i is some vector of parameters that describes the motion (translation or rigid motion) applied to object i. Then, the fact that all objects are packed inside the container is described by the formula

$$\forall x \bigwedge_{i=1}^{n} (F_i(t_i, x) \Rightarrow F(x)) \tag{1}$$

and the fact that the objects are placed nonoverlapping by

$$\forall x \bigwedge_{i \neq j} (F_i(t_i, x) \Rightarrow \neg F_j(t_j, x)) \tag{2}$$

.

The conjunction of both formulas gives a formula ϕ with free variables from $t_1, ..., t_n$. Then $\exists t_1 ... \exists t_n \phi$ is a *Tarski formula* of polynomial size. The truth of such a formula can be decided in *exponential time* by the algorithms of Renegar [26–28] or Canny [7].

For simple kinds of objects and containers, e.g., circles, spheres, convex polyhedra in constant dimension, formulas (1) and (2) can be replaced by formulas without an all-quantifier. So, altogether we obtain a formula within the *existential first order theory of the reals*. By Canny's algorithm [7] the truth value of such formulas can be decided within PSPACE.

For the optimization problem, we have a usually infinite set of possible containers. We assume that elements of this set are specified by some vector r of parameters, so that the corresponding container $F(r, x)$ is described by some algebraic expression in r and x. E.g., For a circle, r will just be its radius, for a triangle, r may consist of the three side lengths. Furthermore, we assume that the *size* $s(r)$ (e.g. area, volume, perimeter) of the container $F(r)$, which is the objective function to be minimized, is given by some algebraic function of r.

Let $\phi(r)$ be the formula indicating that all objects can be packed into the container specified by $F(r, x)$, i.e., the conjunction of formulas (1) and (2) where $F(x)$ is replaced by $F(r, x)$ and r is a set of free variables. Then adding by conjunction

$$\phi(r) \wedge (\forall r' (s(r') < s(r)) \Rightarrow \neg\phi(r')) \tag{3}$$

is a Tarski formula which is true for any r specifying a minimum size container. The decision algorithms for Tarski formulas also determine the assignment of algebraic numbers to r which makes formula (3) true, i.e. they give us the desired solution.

Observe that formula (3) contains an all-quantifier, i.e. is not a formula in the *existential* first order theory of the reals. It is not clear how that quantifier can be avoided. Therefore, for optimization problems of the kind described we can only say that they are in EXPTIME, it is not clear that they are in PSPACE.

So, *in theory*, all packing problems of the kind described, can be solved by running an algorithm, i.e., a computer program, given sufficient time.

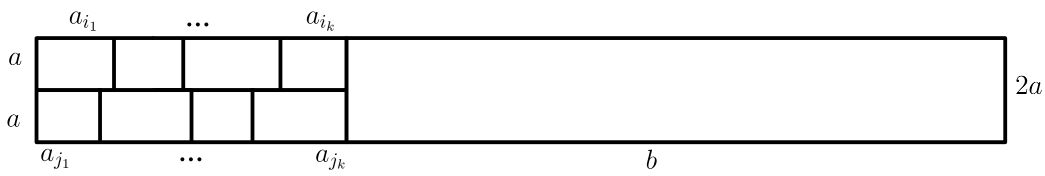

Figure 3: NP-completeness of packing axis-parallel rectangles into an axis-parallel rectangle: perfect packing is only possible, if the set of numbers $a_1, ..., a_n$ can be partitioned into two subsets of equal sums.

Let us consider, for example, the open problem, whether the container in Figure 2 has minimum radius. We can assume that the container is a disk centered at the origin, which leaves 25 variables for the coordinates of the centers of the twelve circles to be packed and the radius of the container. The constraints in formulas (1) and (2) can, in the case of circles, be formulated as quadratic constraints in the variables. It appears, however, that (nonconvex) quadratic programming is computationally a very difficult problem so that even the most efficient computer algebra systems nowadays cannot solve it in general for 25 variables in a reasonable amount of time. Notice that we already observed this discrepancy between computability in theory and really obtaining the correct result in the problem of proving of Kepler's conjecture.

4 NP-hardness

Figure 3 shows a reduction from the NP-complete problem PARTITION to various packing problems. a and b are constants with $a \in (0, 1/2)$ and $b > \sum_{i=1}^{n} a_i$. Even one of the most basic packing problems, namely the decision problem whether a set of axis-parallel rectangles can be packed by translations into a given container which is also an axis-parallel rectangle is, thus, NP-complete (see also [12]).

Observe, that he reduction also works if rotations of the rectangles are allowed. Furthermore, this reduction shows that many decision problems of the form "Can a given set of objects be packed into a given container?" are NP-hard. In fact, the objects and the container may be axis-parallel rectangles, arbitrary rectangles, convex polygons, or arbitrary simple polygons. For all possible combinations and both, translations and rigid motions, the problem is NP-hard.

Likewise, the optimization problem of finding a minimum area container is NP-hard for all combinations of types of objects, containers, and motions mentioned, with the exception of simple polygons as containers.

It does not follow that the decision problem whether a given set of axis-parallel squares can be packed into a given square is NP-hard. That fact, however, was shown by Leung et al. [21] where by a reduction from 3-PARTITION [13] even strong NP-hardness is shown.

One of the few circle packing problems for which an NP-hardness result is known is presented by Demaine et al. [9]. By a sophisticated reduction from 3-PARTITION they show that the problem whether a given set of circles can be packed into a given triangle, a given rectangle, or a given square is NP-hard.

The decision problem whether packing given circles into a given circle is possible (see also Section 2) is, to our knowledge, not known to be NP-hard.

Naturally, the question arises which of these problems are **NP-complete**, i.e., are in NP. In the previous section we argued that the decision versions of all the problems of this section are in PSPACE and the optimization versions are in EXPTIME. Surprisingly, only in one case we can reduce their complexity to NP.

In fact, the decision problem whether a set of axis-parallel rectangles can be packed by translations into a given axis-parallel rectangular container is in NP. Assume without loss of generality that the lower left cormer of the container is the origin. We can assume that, if a packing exists, then there is one where no "gaps" along a complete side of a rectangle in x- or y-direction occur. Otherwise the rectangles could be shifted to close the gaps. Consequently, there is always a packing, if any, where the x-coordinates of the lower left vertices of the rectangles are sums of some widths of the given rectangles. Likewise, the y-coordinates of these vertices are sums of some of the heights. Guessing these sums and verifying whether that placement is a packing gives a polynomial-time nondeterministic algorithm for the problem.

For the other variants of the packing problems in this section membership in NP seems to be open.

Even for the simply stated problem of *Pallet Loading* neither NP-hardness nor membership in NP is known. The input of that problem consists of five positive integers a, b, A, B, n and the question is whether n rectangles of sidelengths a and b can be packed in either axis-parallel orientation into a rectangle of sidelengths A and B (see e.g. http://www.cs.smith.edu/~orourke/TOPP/P55.html#Problem.55). The reason for the unknown status of the problem is that its description is quite compact if the numbers are given in binary and it cannot be excluded that packing patterns are quite complex. Observe, that the considerations of the previous section cannot be applied to this problem, either.

5 Constantly many objects: exact solution

Because of the NP-hardness results we can only hope for finding the optimal solution in polynomial time for a constant number of objects. This problem has been investigated for polygons.

Finding the minimum area convex container for two convex polygons under translation can be done in linear time [20]. Considerably more difficult is finding the optimal container of two convex polygons P and Q under rigid motions. For rectangular containers this was shown by F. Hurtado and the author [2]. For arbitrary convex containers, Tang et al. [31] give an algorithm of runtime $O((n+m)nm)$ where n and m are the numbers of vertices of P and Q, respectively. The algorithm is very elaborate by checking all possible contact configurations of the two polygons.

Park et al. consider the optimal packing of three convex polygons under translations [25]. They show how to construct the minimum area convex container in $O(n^2)$ time by characterizing and investigating all combinatorially equivalent positions three polygons can assume. For each of these, the formula for the area of the convex hull can efficiently be determined from the previous one and the minimum found is returned.

The general case of packing k polygons under translations as well as rigid motions into a given container has been dealt with in a series of articles by Daniels and Milenkovic[8, 22–24]. Using an especially designed technique, linear programming based restriction, they show that all these problems can be solved in time $(nm)^{O(k)}$ where n is the number of edges of the container and m is the number of edges of each polygon. Remarkably, also problems of implementation and applications to real world problems, mostly from apparel industry, are dealt with in this series of papers.

6 Heuristics and Approximate Strip Packing

Because of its significance in practical applications, many heuristic algorithms for packing have been developed and implemented, in particular in the operations research community. The monograph of G. Scheithauer [30] (in German) is a good survey on the techniques used, which include linear programming, backtracking, branch and bound, metaheuristics like simulated annealing or genetic algorithms, and others designed particularly for the problem at hand.

For some cases, those algorithms even have a guaranteed performance concerning the approximation ratio with the optimal solution. More precisely, let $A(I)$ be the value of the objective function for the output that algorithm A produces for problem instance I and $OPT(I)$ the optimal value. We say that A has an *absolute*

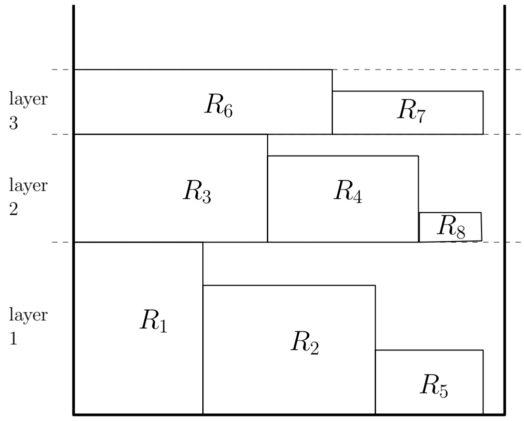

Figure 4: Rectangles packed by FFDH.

performance ratio α iff $A(I) \leq \alpha OPT(I)$. The performance ratio is called *asymptotic*, iff $limsup_{OPT(I)\to\infty} A(I)/OPT(I) = \alpha$. So, in the asymptotic case $A(I)$ may have some additive terms whose ratio to $OPT(I)$ tends to 0 for large enough (and suitably chosen) instances.

For strip packing of rectangles in two dimensions a simple greedy algorithm [19], namely first-fit-decreasing height (FFDH), gives an asymptotic performance bound of $1.7OPT + 1$. Rectangles are sorted by decreasing height, $R_1, R_2, ...$, and packed on "layers" of the strip where each rectangle R_i is packed on the lowest possible layer. If R_i does not fit into any of the existing layers any more, a new layer with the height of R_i is started, see Figure 4.

The best known absolute performance factor for this problem is meanwhile $5/3 + \varepsilon$ for any constant $\varepsilon > 0$ by Harren et al. [16].

Concerning asymptotic approximations, the best result has been found by Jansen and Solis-Oba [18]. In fact, they obtain a so-called APTAS, i.e., for any $\varepsilon > 0$ an asymptotic approximation with multiplicative factor $1 + \varepsilon$.

7 Approximating the Minimum Volume of a Container

7.1 Two-dimensional packing

An algorithm $\mathcal{A}_m$ for approximating a minimum area axis-parallel rectangle as container for a set of n axis-parallel **rectangles under translations** can be derived from an algorithm $\mathcal{A}_{sp}$ for approximating optimal strip-packing as follows (see also [29]).

The width w_{opt} of the optimal container is at least w_{max}, the maximum width of any object and at most $n \cdot w_{max}$. We apply $\mathcal{A}_{sp}$ to strips of width $w_{max}, w_{max}(1 + \varepsilon), ..., w_{max}(1 + \varepsilon)^k$ where k is the smallest number with $w_{max}(1 + \varepsilon)^k \geq n \cdot w_{max}$. The minimum area rectangle (= strip segment) found this way is returned as an approximation for the minimum area container.

In fact, suppose that $\mathcal{A}_{sp}$ has an approximation factor of α. Let h_{opt} be the height of the minimum area container with width w_{opt}. Then $\mathcal{A}_{sp}$ has been applied to a strip of width at most $w_{opt}(1 + \varepsilon)$ and has returned a height of at most αh_{opt}. Therefore, the area of the corresponding rectangle is at most $\alpha h_{opt} w_{opt}(1 + \varepsilon) = \alpha(1 + \varepsilon)A_{opt}$, where A_{opt} is the minimum area for an axis-parallel rectangular container. Since $k = \lceil \log n / \log(1 + \varepsilon) \rceil$ the runtime of $\mathcal{A}_m$ is $O(T(n) \log n \cdot 1/\varepsilon)$ where $T(n)$ is the runtime of $\mathcal{A}_{sp}$.

In short, from an efficient approximation algorithm for strip packing we can obtain an efficient approximation algorithm for the smallest axis-parallel rectangular container which has nearly the same approximation factor. By the strip-packing algorithm of Harren et al. [16], we obtain a $(5/3 + \varepsilon)$-approximation for finding a minimum area axis-parallel rectangle for packing a set of axis-parallel rectangles under translation.

Approximating an optimal axis-parallel rectangular container for "irregular" convex objects, i.e., **convex polygons under rigid motions** can be reduced to the problem of packing axis parallel rectangles under translations, see [32]. In fact, for each polygon we can determine in linear time the enclosing rectangle of largest width. It can be shown that its area is at most twice as large as the one of the polygon. Then, we rotate the enclosing rectangles (including the polygons) so that their widths become horizontal and apply the translational packing algorithm to that set of axis-parallel rectangles. It can be shown that, if that algorithm is chosen suitably, the resulting algorithm for packing polygons gives a factor-5-approximation for the smallest enclosing rectangle.

For arbitrary rectangles under rigid motions this factor can be reduced to 3, see [32].

The problem of approximating a minimum area axis-parallel rectangular container for **convex polygons under translations** appears to be more difficult. De Berg, Knauer, and the author [1] give an algorithm which works as follows:

Let the *height* of a polygon be the vertical distance from its lowest to its highest point, its *spine* the line segment between two such points, and h_{max} and w_{max} the maximum height and width of the polygons, respectively.

First the set of polygons is partitioned into height classes, determined by heights in the intervals $(\alpha^{i+1} h_{max}, \alpha^i h_{max}], i = 0, 1, ...,$ where h_{max} is the maximum height of the polygons and $\alpha \in (0, 1)$ some suitable constant.

Within height class i the polygons are packed consecutively into a strip of height $\alpha^i h_{max}$ ordered by the slopes of their spines, see Figure 5. Each strip is

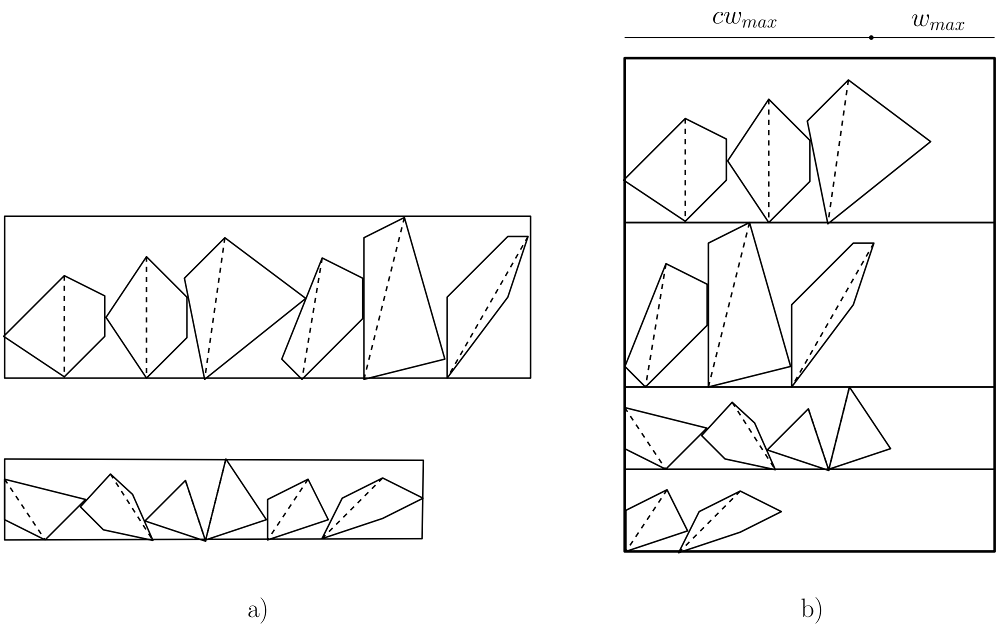

Figure 5: Packing polygons under translations. a) after partitioning by height and sorting by spine slope. b) Resulting axis-parallel container.

partitioned into segments of width cw_{max} where $c \geq 1$ is a suitable constant. Each segment contains the polygons whose leftmost point is within the segment and is extended by a piece of length w_{max} in order to be able to contain the complete polygons. All these segments having equal width $(c+1)w_{max}$ are stacked to form an axis-parallel rectangle which is returned as the solution.

Optimizing parameters α and c yields an approximation factor of 17.449... In practice, this constant is exceedingly high, but in most cases the output can be improved by simple heuristics. Furthermore, the major insight from this algorithm is that problem is constant factor approximable at all, i.e., it lies in complexity class APX.

The algorithm described can also be modified to find a 27-approximation to the smallest area convex container.

7.2 Three-dimensional packing

The analogues of rectangles and convex polygons in three dimensions are (rectangular) cuboids (shortly called "boxes") and convex polyhedra, repectively.

There is some research on **strip packing** of axis-parallel boxes. As was mentioned before, in three dimensional strip packing some rectangular cross section in the x-y-plane of a strip is fixed and the objective is to pack the given set of objects (axis-parallel boxes) by translations into a container with this cross ection and minimum height. The best approximation algorithm with respect to asymptotic approximations was found recently by Jansen and Prädel [17] which has an approximation factor of $3/2 + \varepsilon$. The best abslute approximation is due to Diedrich et al. [10] with an approximation factor of $29/4$.

In her master's thesis [29], see also [3], Scharf investigated three-dimensional packing where the **minimum volume container** is wanted and found approximation algorithms for several variants.

In fact, it turns out that packing **axis parallel boxes by translation** into an approximately minimum volume axis-parallel box can be reduced to strip packing.

In fact, a technique analogous to the one for two dimensions described in Section 7.1 can be used. More precisely, letw_{max} and d_{max} be the maximum width (x-direction, say) and depth (y-direction) of the boxes to e packed. Then, for all strips of widths $w_{max}(1 + \varepsilon)^i$ and depths $d_{max}(1 + \varepsilon)^j$ that lie in the intervals $[w_{max}, nw_{max}]$ and $[d_{max}, nd_{max}]$, respectively, approximately optimal strip heights are found by the strip packing algorithm. The one with the minimum volume is returned.

Thus, if the strip packing algorithm has an absolute approximation factor of α then a minimum volume finding algorithm with approximation factor $(1 + \varepsilon)\alpha$ can be found for any arbitrary $\varepsilon > 0$. Applying this technique to the strip packing algorithm by Diedrich et al. gives a minimum volume approximation algorithm with factor $29/4 + \varepsilon$.

In [3, 29], it is also shown that a similar idea as in the two-dimensional case can be used to find an approximation algorithm for the **packing of a set of arbitrary convex polyhedra** by rigid motions into a minimum volume axis-parallel box. First, to each polyhedron P an enclosing box B is found by choosing two sides of B perpendicular to the diameter of P. Then the projection onto one of these sides is some convex polygon P' and again two further sides of B are chosen perpendicular to the diameter of P'. The remaining sides of B are chosen perpendicular to the first two pairs of sides so that they touch P. Then, it can be shown that the volume of B is at most six times the one of P. If some suitable approximate packing algorithm is chosen to pack all enclosing boxes into an axis-parallel box C, then the volume of C is at most 277.59 times larger than the one of the optimal solution. Again, this is quite a large constant, but at least it shows that the problem is approximable, i.e., in class APX.

If we allow the container to be an arbitrary convex polyhedron the ideas above can be applied to find a 29.135-approximation for cuboids under rigid motion and a 511.37-approximation for convex polyhedra.

It still is an open problem whether finding a minimum volume container (cuboid or convex) for a set of convex polyhedra is in APX, i.e., can be approximated in polynomial time.

References

[1] Helmut Alt, Mark de Berg, and Christian Knauer. Approximating minimum-area rectangular and convex containers for packing convex polygons. In *Algorithms - ESA 2015 - 23rd Annual European Symposium, Patras, Greece, September 14-16, 2015, Proceedings*, pages 25–34, 2015.

[2] Helmut Alt and Ferran Hurtado. Packing convex polygons into rectangular boxes. In *Discrete and Computational Geometry, Japanese Conference, JCDCG 2000, Tokyo, Japan, November, 22-25, 2000, Revised Papers*, pages 67–80, 2000.

[3] Helmut Alt and Nadja Scharf. Approximating smallest containers for packing three-dimensional convex objects. *CoRR*, abs/1601.04585, 2016.

[4] Ulrich Betke and Martin Henk. Finite packings of spheres. *Discrete & Computational Geometry*, 19(2):197–227, 1998.

[5] Ulrich Betke, Martin Henk, and Jörg M Wills. Finite and infinite packings. *J. reine angew. Math*, 453:165–191, 1994.

[6] K. Börözky. *Finite Packing and Covering*. Cambridge University Press, 2004.

[7] John F. Canny. Some algebraic and geometric computations in PSPACE. In *Proceedings of the 20th Annual ACM Symposium on Theory of Computing, May 2-4, 1988, Chicago, Illinois, USA*, pages 460–467, 1988.

[8] Karen L. Daniels and Victor Milenkovic. Multiple translational containment. part i: An approximate algorithm. *Algorithmica*, 19(1/2):148–182, 1997.

[9] Erik D. Demaine, Sándor P. Fekete, and Robert J. Lang. Circle packing for origami design is hard. *CoRR*, abs/1008.1224, 2010.

[10] Florian Diedrich, Rolf Harren, Klaus Jansen, Ralf Thöle, and Henning Thomas. Approximation algorithms for 3d orthogonal knapsack. *J. Comput. Sci. Technol.*, 23(5):749–762, 2008.

[11] L. Fejes Tóth. "Research Problems.". *Periodica Mathematica Hungarica*, 6(2):197–199, 1975.

[12] Robert J. Fowler, Mike Paterson, and Steven L. Tanimoto. Optimal packing and covering in the plane are NP-complete. *Inf. Process. Lett.*, 12(3):133–137, 1981.

[13] M. R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

[14] Peter Gritzmann and Jörg M. Wills. Finite packing and covering. In *Handbook on Convex Geometry*. North Holland, 1993.

[15] T. Hales, M. Adams, G. Bauer, Dang Tat Dat, J. Harrison, H.L. Truong, C. Kaliszyk, V. Magron, S. McLaughlin, Nguyen Tat Thang, Nguyen Quang Truong, T. Nipkow, S. Obua, J. Pleso, J. Rute, A. Solovyev, Ta Thi Hoai An, Tran Nam Trung, Trieu Thi Diep, J. Urban, Vu Khac Ky, and R. Zumkeller. A formal proof of the Kepler conjecture. 2014.

[16] Rolf Harren, Klaus Jansen, Lars Prädel, and Rob van Stee. A $(5/3 + \epsilon)$-approximation for strip packing. *Comput. Geom.*, 47(2):248–267, 2014.

[17] Klaus Jansen and Lars Prädel. A new asymptotic approximation algorithm for 3-dimensional strip packing. In *SOFSEM 2014: Theory and Practice of Computer Science - 40th International Conference on Current Trends in Theory and Practice of Computer Science, Nový Smokovec, Slovakia, January 26-29, 2014, Proceedings*, pages 327–338, 2014.

[18] Klaus Jansen and Roberto Solis-Oba. Rectangle packing with one-dimensional resource augmentation. *Discrete Optimization*, 6(3):310–323, 2009.

[19] Edward G. Coffman Jr., M. R. Garey, David S. Johnson, and Robert Endre Tarjan. Performance bounds for level-oriented two-dimensional packing algorithms. *SIAM J. Comput.*, 9(4):808–826, 1980.

[20] Hyun-Chan Lee and Tony C. Woo. Determining in linear time the minimum area convex hull of two polygons. *IIE Transactions*, 20(4):338–345, 1988.

[21] Joseph Y.-T. Leung, Tommy W. Tam, C. S. Wong, Gilbert H. Young, and Francis Y. L. Chin. Packing squares into a square. *J. Parallel Distrib. Comput.*, 10(3):271–275, 1990.

[22] Victor Milenkovic. Translational polygon containment and minimal enclosure using linear programming based restriction. In *STOC*, pages 109–118, 1996.

[23] Victor Milenkovic. Multiple translational containment. part ii: Exact algorithms. *Algorithmica*, 19(1/2):183–218, 1997.

[24] Victor Milenkovic. Rotational polygon containment and minimum enclosure. In *Symposium on Computational Geometry*, pages 1–8, 1998.

[25] Dongwoo Park, Sang Won Bae, Helmut Alt, and Hee-Kap Ahn. Bundling three convex polygons to minimize area or perimeter. *Comput. Geom.*, 51:1–14, 2016.

[26] James Renegar. On the computational complexity and geometry of the first-order theory of the reals, part I: introduction. preliminaries. the geometry of semi-algebraic sets. the decision problem for the existential theory of the reals. *J. Symb. Comput.*, 13(3):255–300, 1992.

[27] James Renegar. On the computational complexity and geometry of the first-order theory of the reals, part II: the general decision problem. preliminaries for quantifier elimination. *J. Symb. Comput.*, 13(3):301–328, 1992.

[28] James Renegar. On the computational complexity and geometry of the first-order theory of the reals, part III: quantifier elimination. *J. Symb. Comput.*, 13(3):329–352, 1992.

[29] Nadja Scharf. Approximating smallest containers for packing three-dimensional convex objects. Master's thesis, Freie Universität Berlin, 2015.

[30] G. Scheithauer. *Zuschnitt- und Packungsoptimierung: Problemstellungen, Modellierungstechniken, Lösungsmethoden*. Studienbücher Wirtschaftsmathematik. Vieweg+Teubner Verlag, 2008.

[31] Kai Tang, Charlie C. L. Wang, and Danny Z. Chen. Minimum area convex packing of two convex polygons. *Int. J. Comput. Geometry Appl.*, 16(1):41–74, 2006.

[32] L. von Niederhäusern. Packing polygons. Master's thesis, EPFL Lausanne, FU Berlin, 2014.