

PROFINITE TECHNIQUES FOR PROBABILISTIC AUTOMATA

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Abstract

The model of (reactive) probabilistic automata was introduced by Rabin in 1963 [10], generalising non-deterministic automata by assigning probabilities to transitions. Such an automaton associates with a word a value between 0 and 1, which is the probability that the run is accepted.

This extended abstract presents recent progress on the value 1 problem for probabilistic automata, which asks whether given a probabilistic automaton, there exist words accepted with arbitrarily high probability.

Since its introduction by Gimbert and Oualhadj in 2010 [8], the value 1 problem has been studied at great depth, leading to the development of new tools of algorithmic, algebraic, and topological nature. We report on a recent paper which introduces a topological approach called the prostochastic theory for understanding the value 1 problem [6, 7].

1 The value 1 problem: motivations and context

The synthesis problem for partially observable systems. A partially observable Markov decision process (POMDP) is a stochastic system whose evolution depends on the actions of a controller having only a partial observation on the evolution of the system. This model appears in various fields such as operational research, artificial intelligence and motion planning in robotics. Developing formal methods to analyse such systems is a major challenge, which attracted a lot of attention in the past ten years.

The *synthesis problem* asks, given a POMDP and a specification, whether there exists a controller for this POMDP ensuring the specification. Unfortunately, this problem is in general undecidable; as shown by Gimbert and Oualhadj it remains undecidable under two restrictions [8]:

- the controller is blind, *i.e.* does not observe anything about the evolution,
- the objective is to satisfy the specification with arbitrarily high probability.

The setting of blind controllers corresponds to the model of probabilistic automata. They were introduced by Rabin [10] as an extension of non-deterministic automata over finite words. A probabilistic automaton \mathcal{A} assigns to every finite word u a value in $[0, 1]$, which is the probability that a run ends up in an accepting state, denoted $\mathcal{P}_{\mathcal{A}}(u)$.

The value 1 problem takes as input a probabilistic automaton \mathcal{A} and asks whether there exists a sequence of words $(u_n)_{n \in \mathbb{N}}$ such that $\lim_n \mathcal{P}_{\mathcal{A}}(u_n) = 1$. Equivalently, is it true that for all $\varepsilon > 0$, there exists a word u such that $\mathcal{P}_{\mathcal{A}}(u) \geq 1 - \varepsilon$? In such case, we say that \mathcal{A} has value 1.

Related works. Over the past years, different restrictions on the class of probabilistic automata have been introduced in order to obtain decidability results. The first subclass which was introduced specifically to decide the value 1 problem are the \sharp -acyclic automata by Gimbert and Oualhadj [8]. Later on Chatterjee and Tracol [3] introduced structurally simple automata, which are probabilistic automata satisfying a structural property (related to the decomposition-separation theorem from probability theory), and proved that the value 1 problem is decidable for structurally simple automata. Chadha, Sistla and Viswanathan introduced the subclass of hierarchical automata [2], and showed that this restriction allows one to recover decidability results.

The Markov monoid algorithm. Gimbert, Oualhadj and the author introduced an algorithm for the value 1 problem called the Markov monoid algorithm [5]. It is based on the notion of stabilisation monoids defined by Colcombet in the study of regular cost functions [1]. A stabilisation monoid is a set equipped with an associative product and a unary operation \sharp called stabilisation. The intuitive meaning of e^\sharp for an element e is $\lim_n e^n$: stabilisation monoids have a built-in notion of limits.

The Markov monoid of a probabilistic automaton generalises the transition monoid for non-deterministic automata by adding the stabilisation operation. Intuitively, each element of the Markov monoid represents the action of a sequence of words. The Markov monoid algorithm takes as input a probabilistic automaton, computes the Markov monoid of this automaton and looks for value 1 witnesses.

Since the value 1 problem is undecidable, the Markov monoid algorithm does not solve it for the whole class of probabilistic automata. However, we introduced the class of probabilistic leaktight automata, and proved the following theorem, using the factorisation forest theorem of Simon [11].

Theorem ([5]). *The Markov monoid algorithm solves the value 1 problem for probabilistic leaktight automata.*

Moreover, this result extends all the previous results in this direction.

Theorem ([4]). *The classes of $\#$ -acyclic, structurally simple and hierarchical probabilistic automata are strictly subsumed by the class of probabilistic leak-tight automata.*

The combination of these two results imply that the Markov monoid algorithm subsumes all other algorithms for the value 1 problem. The aim of the prostochastic theory is to develop topological tools for analysing this algorithm.

2 The prostochastic theory

Profinite theory. The profinite theory is a deep mathematical theory originating from topology. It has been developed in automata theory by Almeida, Pin, Weil and others, see for instance [9]. In this context, it consists in constructing the topological completion of the set of finite words. In other words, it allows one to define the notion of converging sequences of finite words and their limits.

The prostochastic theory follows the same approach, generalising it to probabilistic automata. In particular, we construct the topological completion of the set of finite words, with respect to all probabilistic automata. We proved the following result, which reformulates the value 1 problem over finite words as the emptiness problem over prostochastic words.

Theorem ([6, 7]). *Let \mathcal{A} be a probabilistic automaton. Then:*

- \mathcal{A} has value 1 if, and only if,
- there exists a prostochastic word accepted by \mathcal{A} .

Convergence speeds. Our motivations for introducing the prostochastic theory is to formalise the notion of convergence speeds. Indeed, analysing the undecidability proof reveals that the constructed probabilistic automata create two competing converging speeds making the combined behaviour hard to describe, and in particular not taken into account by the Markov monoid algorithm.

These behaviours only arise with non-polynomial sequences. Informally, we say that the sequence $((a^n b)^n)_{n \in \mathbb{N}}$ is polynomial, while the sequence $((a^n b)^{2^n})_{n \in \mathbb{N}}$ is not, since n and 2^n are not polynomial related. We proved that the Markov monoid algorithm captures exactly all polynomial sequences, which is formalised in the following theorem.

Theorem ([6, 7]). *Let \mathcal{A} be a probabilistic automaton. Then:*

- the Markov monoid algorithm answers “YES” on input \mathcal{A} , if, and only if,
- there exists a polynomial sequence $(u_n)_{n \in \mathbb{N}}$ such that $\lim_n \mathcal{P}_{\mathcal{A}}(u_n) = 1$.

This theorem precisely characterises the computations of the Markov monoid algorithm. Its proof is rather technical as it requires to obtain precise bounds on convergence phenomena of sequences of Markov chains. The prostochastic theory is a language to formalise this proof, meaning that it provides a set of definitions and notions that are used to formulate the statements and the proofs of this theorem.

Formally, we define an ω operator echoing the profinite theory for classical automata, and obtain the set of polynomial prostochastic words as induced by the set of ω -terms. We prove the following characterisation theorem, equivalent to the previous one.

Theorem ([6, 7]). *Let \mathcal{A} be a probabilistic automaton. Then:*

- *the Markov monoid algorithm answers “YES” on input \mathcal{A} , if, and only if,*
- *there exists a polynomial prostochastic word accepted by \mathcal{A} .*

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