On the Computational Power of Simple Dynamics

Extended Abstract

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Simple interaction rules are widespread in Nature and have unsurprisingly been a central object of study in many sciences. In physics, for example, the investigation of interacting particle systems has played a central role in establishing statistical mechanics as a fundamental theory connecting the atomistic viewpoint to macroscopic phenomena [22]. More recently, theoretical computer scientists have also started to contribute[1] to this general endeavor by analyzing simple interaction rules from a *computational* point of view, with the initial motivation of developing new algorithmic principles for the design of *lightweight* algorithms, called dynamics, which are characterized by a level of simplicity comparable to that of interacting particle systems. As an early example, [19] analyzed the famous Voter model [22] (here, Voter dynamics), as a proportionate-consensus dynamics for distributed system. However, a long observed fact is that very few such processes seem amenable to be treated analytically [3]. This naturally raises the question of whether it is possible to make advances towards a rigorous theory regarding dynamics.

This dissertation developed as an inquiry in such direction. It consists of two parts. The first part is concerned with the solution of foundational distributed computing problems through the use of dynamics[2], by advancing the rigorous analysis for many of them. The second part is concerned with the application of these dynamics to inquire into the minimal model assumptions which still enable some primitive distributed computing problems to be efficiently solved. This is achieved by mainly considering the problem of information spreading when communication is affected by noise, and when a self-stabilizing solution is required.

Informally, dynamics are rules to update an agent's state according to a function which

[1] According to certain historical perspectives, one could rather talk about a recent **revival** of attention from computer scientists: In mathematics and theoretical computer science, the study of basic algorithmic processes on very simple distributed systems can be traced back to the study of cellular automata and Conway's Game of Life formulated in the '70s. There are, however, important differences which distinguish the approach discussed here from these previous ones. We defer a discussion on this point to a forthcoming survey on dynamics.

[2] To simplify exposition, this abstract does not reflect the same topic order of the dissertation chapters.

is invariant with respect to time, graph topology, and the identity of an agent's neighbors, and whose arguments are only the agent's current state and those of its neighbors. Famous examples are the aforementioned Voter dynamics [22], 2-Median [13], 2-Choice [12], 3-Majority [5, 6], Undecided-State [4] and Averaging [7]. In the following, we summarize the contribution of the dissertation in understanding the latter three.

Undecided-State dynamics

The work [19] mentioned above motivated the investigation of efficient dynamics for the fundamental problem of majority consensus [5, 12], in which the goal is to rapidly converge to the value initially hold by the majority of agents. In particular, an open problem was to exhibit a dynamics whose convergence time is sublinear with respect to the number of different opinions initially hold by agents. In the first part of the dissertation we provide a careful and general analytical treatment of the evolution of the Undecided-State dynamics[3] (in the binary opinion case, also known as 3-States dynamics), on complete topologies, which has been for long a captivating candidate for rapidly solving the majority consensus problem [1, 24]. The dynamics is defined as follows: at each round, each agent observes a random neighbor; if the neighbor is in the same state, the agent does nothing; otherwise, if the neighbor is in a different state, the agent switches to a special *undecided* state; once undecided, an agent just copies the state of the next agent that it observes.

In the dissertation, we survey previous analyses of dynamics for majority consensus which provide upper bounds on the convergence time: essentially all of these bounds are a function of a simple parameter of the initial configuration, called *bias*, namely a sufficiently large difference between the opinion size of the majority opinion and that of the second-majority one. Rather surprisingly, we show that the convergence time of the Undecided-State dynamics turns out to depend on the initial configuration in a much more sophisticated way: the convergence time is proportional to the *monochromatic distance* $\sum_i (c_i/c_m)^2$, i.e. the sum of the squares of the opinion sizes c_i, divided by the square of the majority opinion size c_m (where by *opinion size* we mean the number of agents initially supporting that opinion). The latter quantity is indeed linear with respect to the number of different initial opinions when the opinion sizes are almost uniform, and gets closer to 1 as the opinion sizes have a very skew distribution.

In subsequent works, our analysis of the Undecided-State dynamics was used as a basic building block to design time-optimal majority consensus protocols [17, 18, 14]. By combining the Undecided-State dynamics with a simple random-walk-based subroutine which allows agents to sample other agents of the network, we also extend the applicability of our results to expander graphs. The latter sampling procedure motivates also the investigation of a *repeated* version of the classical balls-into-bins experiment in discrete probability where, at each round, a ball is extracted from each non-empty bin and throw again uniformly at random in a bin. Through a coupling-with-high-probability argument, we show that the repeated n-balls-in-n-bins process exhibits a maximum load of logarithmic order[4] (in contrast to the classical balls-into-bins experiment whose maximum load is of order

[3] This subset of results has appeared in [4].

[4] This subset of results has appeared in [10].

$\Theta(\log n / \log\log n)$), which implies, in particular, that parallel random walks in the Gossip model have low congestion.

3-Majority dynamics

As already mentioned, essentially all previous work which tackled the majority consensus problem, including the aforementioned analysis of the Undecided-State dynamics, crucially relies on the assumption of a sufficient *bias* in the initial configuration. This constraint on the initial configuration has been the main obstacle in using majority dynamics for the more general *valid consensus* problem, in which the system is required to converge to consensus on any opinion that was present in the initial configuration. The main technical challenge one has to face in removing the initial bias assumption is to show that a *symmetry-breaking phase* takes place in reasonable time: starting from the uniform configuration in which all opinion sizes are equal, the opinion sizes rapidly change, resulting in the process making progress toward a consensus configuration. In the majority consensus problem the natural approach generally consists in showing that the difference between the opinion size of the majority opinion and that of any other one progressively gets larger and larger. Without the bias assumption, it turns out that such approach can hardly work. Consider, for example, the 3-Majority dynamics: at each round, each agent looks at three neighbors sampled independently and uniformly at random with replacement and update its state with the *majority* among the observed neighbors' states, breaking ties uniformly at random. There are many possible initial configurations in which there is a subset of large opinions (an "oligarchy"), which have roughly the same size and "compete" from round to round for which of them is the majority one, resulting in the majority opinion to be a random variable which changes from one round to the next. In this work, we first provide a tight analysis the 3-Majority dynamics[5], together with an upper bound on the improvement that one can get by considering h-Majority dynamics, and a proof of optimality of 3-Majority as the best 3-input dynamics for majority consensus (i.e., any other dynamics would fail to compute the majority on some initial configurations). We then provide a new analysis technique whose key is to look at the evolution of *small* opinions sizes[6]: we examine the 3-Majority dynamics and show that the process has a strong tendency in reducing the size of at least one opinion and, as soon as such opinion drops below a certain large threshold, it rapidly disappears. We further show that this approach can be easily adapted to prove bounds on the fault tolerance of the 3-Majority dynamics against a dynamic adversary [13]. Overall, the analysis presented in this dissertation is the first *unconditional* analysis of a fast majority consensus dynamics, which is also fault-tolerant against moderately powerful adversaries.

Averaging dynamics

The previous results showed that important distributed computing problems can be efficiently solved by employing simple dynamics. We then address the following question: Is there a problem, that dynamics can efficiently solve, which is non-trivial even in a central-

[5] This subset of results has appeared in [5].

[6] This subset of results has appeared in [6].

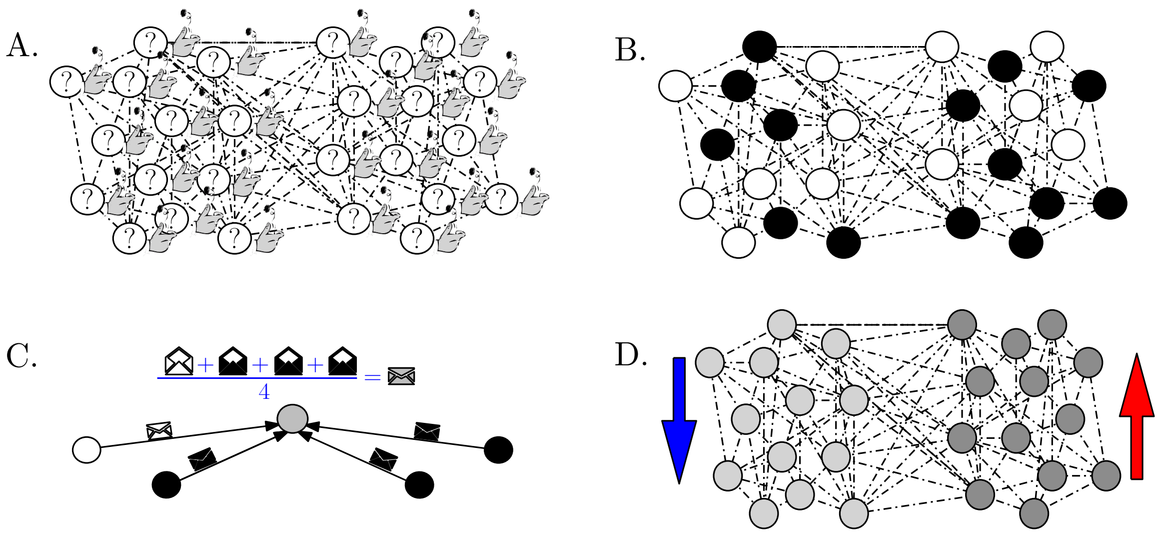

Figure 1: A diagram of the averaging protocol for distributed reconstruction. **A-B)** Each agent initially flips a fair coin and assigns itself a value +1 or -1 accordingly. **C)** At each subsequent round, each agent updates its value with the *average* of the values of its neighbors. **D)** At each round, each agent also checks whether its value increased compared to the previous round: if yes, the agent declares itself *blue*, otherwise it declares itself *red*. After a logarithmic number of rounds, this blue-red labeling quickly stabilizes and provides the requested reconstruction.

ized setting? This work provides an affirmative answer by looking at the classical problem of clustering, formalized as a minimum bisection problem.

We consider a family of *clustered graphs* where each node belongs to either one of two equal-size communities, and has more neighbors inside its community than outside it. We ask whether a simple algorithm can *reconstruct* the two communities by assigning binary labels to nodes. We show that the following algorithm achieves the goal[7]: At the outset, each agent flips a fair coin and assigns itself a value +1 or -1 accordingly; Then, at each subsequent round, each agent updates its value with the *average* of those of its neighbors; Additionally, at each round, each agent declares itself *blue* if its value increases compared to the previous round, *red* otherwise. After a logarithmic number of rounds, the blue-red labeling stabilizes and provides the requested reconstruction, either exactly or approximately, depending on the alignment between the space of the first two eigenvectors of the graph and that of the indicator vectors of the communities, and on the ratio between the second largest eigenvalue and the third one in absolute value. In fact, when a regularized version of clustered graphs is considered, the reconstruction is exact and we also show that a natural extensions of the above dynamics perform community detection also with multiple communities.

We then prove that the Averaging dynamics also succeeds with high probability in performing the reconstruction for an almost-optimal range of parameters on graphs generated according to the famous stochastic block model. In the stochastic block model, nodes are

[7] This subset of results has appeared in [7].

divided in two *communities* and, for each pair of nodes in the same community, the edge connecting them is present with probability p, while for each pair of nodes in different community the corresponding edge is present with probability $q < p$. A series of previous works has shown that, on the stochastic block model, reconstructing the planted partition can be done if and only if the model's parameter satisfy a precise phase-transition condition [20, 23, 2, 11], namely $pn - qn > \sqrt{2(p+q)n}$. We further show that, in the regularized version of the previous random graph model, known as *regular stochastic block model*, the Averaging dynamics perform an *exact* reconstruction for the whole range of model parameters which is necessary and sufficient for the reconstruction to be possible [8].

Our analysis of the Averaging dynamics is based on a combination of spectral-linear algebraic techniques and concentration of probability arguments. In particular, we first prove a more general result on deterministic graph classes; then, leveraging on matrix Chernoff bounds and matrix perturbation arguments, we extend the analysis to stochastic block models. Besides its significance as a proof of concept for the computational power of simple dynamics, from a spectral graph theory perspective, this work improves the state of the art on distributed spectral clustering by showing that the power method for computing the second eigenspace can be run implicitly, i.e. without having the agents compute the explicit values of the entries of the second eigenvector. This implies that the blue-red labeling rule can converge in a time polynomially smaller than the mixing time of the graph. In particular, since the convergence time of previous approaches to the use of spectral clustering in a distributed setting were at least as large as the mixing time of the graph, we provide the first protocol which overcomes this limitation.

Noisy information spreading

In the second part of the dissertation we consider two generalizations of the fundamental problem of information spreading (also known as broadcast or rumor spreading), in which the system's goal is to disseminate an information from an initial source agent to the whole system, and related problems such as majority consensus. In doing so, we aim at investigating the minimal model assumptions which still enable the information spreading problem to be efficiently solved, despite limited, chaotic and anonymous communication. The first scenario considers the information spreading problem in the presence of communication noise: while error-correcting codes are efficient methods for handling noisy communication channels in the context of technological networks, such elaborate methods differ a lot from the unsophisticated way biological entities are supposed to communicate. Yet, it has been shown in [15] that complex coordination tasks such as information spreading and majority consensus can plausibly be achieved in biological systems subject to noisy communication channels, where every message transferred through a channel remains intact with almost-uniform probability, without using coding techniques.

The result of [15] is a substantial step towards a better understanding of the way biological entities may cooperate. It has nevertheless been established only in the case of 2-valued opinions, i.e. when information spreading aims at broadcasting a single-bit opinion to all agents, and majority consensus aims at leading all agents to adopt the single-bit opinion that was initially present in the system with (relative) majority. In this work[8], we extend these results to k-valued opinions, for any $k \geq 2$. Our extension requires to address a series of important issues, some conceptual, others technical. We had to entirely revisit the notion of noise, for handling channels carrying k-valued messages. In fact, we precisely characterize the type of noise patterns for which plurality consensus is solvable. Also, a key result employed in the bi-valued case in [15] is an estimate of the probability of observing the most frequent opinion from observing the mode of a small sample. We generalize this result to the multivalued case by providing a new analytical proof for the bivalued case, based on a relation between the cumulative probability function of the binomial distribution and the cumulative density function of the beta distribution. Our new bound is amenable to be extended, by induction, to the multivalued case. Finally, interactions among agents in the considered uniform gossip model do not lead to independent events to which one can directly apply standard concentration of probability tools in order to prove bounds with high probability. To address the latter technical issue, we develop a general framework based on Poisson approximations. Both the latter framework on the use of the relation between the Binomial and Beta distributions are of independent interest.

Self-stabilizing information spreading

In the last part of the dissertation we bring together, for the first time, two important distributed computing concepts: the information spreading problem and the notion of self-stabilization, in which the system is required to converge to a correct state regardless of its initial configuration, which can be an arbitrary combination of the agents' possible states. Notably, despite the fact that both concepts have individually been thoroughly investigated, no substantial work has been devoted to the problem of self-stabilizing information spreading[9]. In fact, our study of this problem is motivated by the observation of biological phenomena: in many biological system, the individuals exhibit a highly coordinated behavior by collectively aligning their actions with that of few specific *source* members of the group, in a dynamic way which rapidly adapts to changes of the sources' behavior. Importantly, the information available to individuals in biological systems is often extremely limited. Hence, we consider the self-stabilizing information spreading problem in the dynamic and unstructured setting of the uniform pull model, in which at each discrete time-step each agent can observe the message displayed by another agent chosen uniformly at random. In particular, we are interested in solving the problem rapidly, i.e. with a convergence time polylogarithmic in the size of the network, and with messages having the smallest possible size.

Observe that, when the topology is stable, bounds on the message size are not significant, since a larger message can be broken in smaller messages and transmitted via pipelining. When the topology is unstable and changes over time, the latter approach doesn't work anymore. Thus, minimizing the message size becomes a challenge in the context of dynamic networks. We provide an almost-logarithmic time algorithm which solves the

[8] This subset of results has appeared in [16].

[9] We note that, in order for the problem to be well defined, a *promise-problem* version of self-stabilization has to be considered, in which it is guaranteed the existence of sources in the initial configuration, according to the problem specifications.

self-stabilizing information spreading problem employing only 3 bits per interaction[10]. To achieve that, we first show that the self-stabilizing information spreading problem can be easily solved in logarithmic time if the agents share a synchronized clock of logarithmic length, with an overhead on the message size of one bit only. We thus turn our attention to the self-stabilizing clock synchronization problem. Clock synchronization has been a foundational research topic in the theory of distributed computing on classical communication models. However, not much attention has been devoted to the case of dynamic networks where agents have no control on with whom they interact, and message size is severely limited. We show how to construct a fast, 3-bit clock synchronization protocol by designing a general transformer which can simulate a certain class of protocols in such a way that reduces *exponentially* the message size, by paying only a logarithmic slowdown in the convergence time. Hence, starting from a simple clock synchronization dynamics, we recursively apply our transformer to obtain the aforementioned 3-bit clock synchronization protocol, and ultimately the self-stabilizing information spreading protocol (without paying any additional bit on the message size). In a nutshell, this work provide the first self-stabilizing information spreading protocol in the uniform pull model, by introducing a general scheme for reducing the message size of many distributed protocols. It thus initiates the study of information spreading from the point of view of self-stabilization, which is a crucial notion of fault-tolerance in computer science.

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