# Hypergraphic Degree Sequences are Hard

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**Abstract**

In their celebrated 1960 paper Erdős and Gallai give an effective characterization of degree sequences of graphs. The analog problem for 3-hypergraphs has been open ever since. We solve it by showing that deciding degree sequences of 3-hypergraphs is NP-complete.

A *k-hypergraph on* $[n]$ is a subset $H \subseteq \{0,1\}^n_k := \{x \in \{0,1\}^n : \|x\|_1 = k\}$. The *degree sequence of* $H$ is the vector $d = \sum H := \sum\{x : x \in H\}$. We consider the following decision problem: given $k$ and $d \in \mathbb{Z}^n_+$, is $d$ the degree sequence of some hypergraph $H \subseteq \{0,1\}^n_k$? For $k = 2$, that is for graphs, the celebrated work of Erdős and Gallai [3, 1960] implies that $d$ is a degree sequence of a graph if and only if $\sum d_i$ is even and, permuting $d$ so that $d_1 \geq \cdots \geq d_n$, the inequalities $\sum_{i=1}^{j} d_i - \sum_{i=l+1}^{n} d_i \leq j(l-1)$ hold for $1 \leq j \leq l \leq n$, yielding a polynomial time algorithm. For $k = 3$ the problem has been open ever since, was formally posed over 30 years ago by Colbourn, Kocay, and Stinson [1, 1986, Problem 3.1], and was recently solved by Deza, Levin, Meesum, and Onn [2, 2018].

Here is the statement and its short proof.

**Theorem** *It is NP-complete to decide if* $d \in \mathbb{Z}_+^n$ *is the degree sequence of a* 3*-hypergraph.*

*Proof.* The problem is in NP since if $d$ is a degree sequences then a hypergraph $H \subseteq \{0,1\}_3^n$ of cardinality $|H| \leq \binom{n}{3} = O(n^3)$ can be exhibited and $d = \sum H$ verified in polynomial time.

We consider the following three decision problems where $\mathbf{1}$ denotes the all-ones vector.

(1) Given $a \in \mathbb{Z}_+^n$, $b \in \mathbb{Z}_+$ with $3\mathbf{1}a = nb$, is there an $F \subseteq \{x \in \{0,1\}_3^n : ax = b\}$ with $\sum F = \mathbf{1}$ ?

(2) Given $w \in \mathbb{Z}^n$, $c \in \mathbb{Z}_+^n$ with $wc = 0$, is there a $G \subseteq \{x \in \{0,1\}_3^n : wx = 0\}$ with $\sum G = c$ ?

(3) Given $d \in \mathbb{Z}_+^n$, is there an $H \subseteq \{0,1\}_3^n$ with $\sum H = d$ ?

Problem (1) is the so-called 3-*partition problem* which is known to be NP-complete [4]. First we reduce (1) to (2). Given $a, b$ with $3\mathbf{1}a = nb$, let $w := 3a - b\mathbf{1}$ and $c := \mathbf{1}$. Then $wc = 0$. Now, for any $x \in \{0,1\}_3^n$ we have $wx = 3ax - b\mathbf{1}x = 3(ax - b)$ so $x$ satisfies $ax = b$ if and only if $wx = 0$. So the answer to (1) is YES if and only if the answer to (2) is YES. Second we reduce (2) to (3). Given $w, c$, with $wc = 0$, define $d := c + \sum S_+$, where $S_\sigma := \{x \in \{0,1\}_3^n : \mathrm{sign}(wx) = \sigma\}$ for $\sigma = -, 0, +$. Suppose there is a $G \subseteq S_0$ with $\sum G = c$. Then $H := G \cup S_+$ satisfies $\sum H = d$. Suppose there is an $H \subseteq \{0,1\}_3^n$ with $\sum H = d$. Then

$$w \sum S_+ = w(c + \sum S_+) = w \sum H = w \sum (H \cap S_-) + w \sum (H \cap S_0) + w \sum (H \cap S_+)$$

which implies $H \cap S_- = \emptyset$ and $H \cap S_+ = S_+$. Therefore $G := H \cap S_0$ satisfies $\sum G = \sum H - \sum S_+ = c$. So the answer to (2) is YES if and only if the answer to (3) is YES. □

# References


[1] Colbourn, C.J., W.L. Kocay, W.L., Stinson, D.R.: Some NP-complete problems for hypergraph degree sequences. Discrete Applied Mathematics 14:239–254 (1986)

[2] Deza, A., Levin, A., Meesum, S.M., Onn, S.: Optimization over degree sequences. SIAM Journal on Discrete Mathematics 32:2067–2079 (2018)

[3] Erdős, P., Gallai, T.: Graphs with prescribed degrees of vertices (in Hungarian). Matematikai Lopak 11:264–274 (1960)

[4] Garey, M.R., Johnson, D.S.: Computers and Intractability. Freeman (1979)