The Algorithmics Column

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Hadwiger's Conjecture for some hereditary classes of graphs: A survey

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Abstract

The famous Hadwiger's Conjecture aims to understand reasons for a graph to have a large chromatic number. This is a survey of graph classes for which Hadwiger's conjecture is known to hold.

Keywords: Hadwiger's Conjecture, graph coloring, minor, hereditary graph class, cap-free graph.

1 Introduction

In this paper, all graphs are finite and simple. A graph G is *t-colorable* if there is a function $c : V(G) \longrightarrow \{1, \ldots, t\}$, such that for every edge uv of G, $c(u) \neq c(v)$. The *chromatic number* of G, denoted $\chi(G)$, is the minimum number t for which G is t-colorable. Graph G is called *t-chromatic* if $t = \chi(G)$. A *minor* of a graph G is obtained from a subgraph of G by contracting edges. As usual, the complete graph on t vertices is denoted by K_t. One of the most famous open problems in graph theory is Hadwiger's Conjecture (HC), made in 1943 [27]:

Hadwiger's Conjecture: *For every $t \geq 0$, every graph with no K_{t+1} minor is t-colorable. (Equivalently: any t-chromatic graph has a K_t minor.)*

For a fixed nonnegative integer t, let HC(t) be the statement: *every graph with no K_{t+1} minor is t-colorable.* Observe that if a graph has no K_2 minor, then it

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is edgeless, and hence 1-colorable; and if a graph has no K_3 minor, then it has no cycles, and hence is 2-colorable. Hadwiger [27] proved HC(3), and in 1952 Dirac [23] in fact showed that every graph of minimum degree at least 3 has a K_4 subdivision (i.e. every graph that has no K_4 subdivision is 3-colorable). In 1937, Wagner [52] showed that HC(4) is equivalent to the Four Color Theorem, which was proved in 1977 by Appel and Haken [2, 3]). In 1993, Robertson, Seymour and Thomas [45] proved HC(5), using the Four Color Theorem (they proved that a contraction-critical 6-chromatic graph $G \neq K_6$ has a vertex x such that $G \setminus x$ is planar, and since planar graphs are 4-colorable by the Four Color Theorem, it follows that G is 5-colorable). For $t \geq 6$, HC(t) remains open. Toft [49], Seymour [47] and Kawarabayashi [31] have written excellent surveys on Hadwiger's Conjecture. In this paper we survey hereditary graph classes for which HC holds, focusing on their structural properties that lead to the proof of HC.

1.1 Terminology and notation

We say that a graph G *contains* a graph F if F is isomorphic to an induced subgraph of G. A graph G is *F-free* if it does not contain F, and for a family of graphs $\mathcal{F}$, G is *$\mathcal{F}$-free* if G is F-free for every $F \in \mathcal{F}$. A class of graphs is *hereditary* if it is closed under isomorphism and taking induced subgraphs. It is not hard to see that a class $\mathcal{G}$ is hereditary if and only if there exists a family $\mathcal{H}$ such that $\mathcal{G}$ is precisely the class of $\mathcal{H}$-free graphs.

A *hole* is a chordless cycle with at least four vertices. A hole is *even* (respectively, *odd*) if it has an even (respectively, odd) number of vertices. An *n-hole* is a hole on n vertices. An *antihole* is a complement of a hole.

The graph C_n is the hole on n vertices. The graph P_n is the path on n vertices (that is, the path of length $n-1$). K_n is the complete graph on n vertices, and $K_{m,n}$ is the complete bipartite graph with m vertices in one part of the bipartition and n vertices in the other. K_3 is also referred to as a *triangle*.

Some small graphs are defined in Figures 1-3. The generalized star $S_{i_1,\dots,i_r}$ is obtained from vertex-disjoint paths of length $i_1, \dots, i_r$ by taking one end-vertex of each path and identifying these into a single vertex. The *claw* is $K_{1,3}$ or equivalently $S_{1,1,1}$. The *chair* or *fork* is $S_{1,1,2}$, the *E* is $S_{1,2,2}$, the *cross* is $S_{1,1,1,2}$ and the *trident* is $S_{1,1,2,2}$.

We subdivide an edge uv in a graph G by deleting it and replacing it by a path u, w, v where w is a new vertex. A *subdivision* of G is obtained by subdividing edges repeatedly. An *induced subdivision of* K_4, denoted ISK_4, is an induced subgraph which is isomorphic to a subdivision of K_4.

The following graphs are pictured in Figure 4. A *theta* is any subdivision of $K_{2,3}$; in particular, $K_{2,3}$ is a theta. A *pyramid* is any subdivision of K_4 in which one triangle remains unsubdivided, and of the remaining three edges, at least two edges are subdivided at least once. A *prism* is any subdivision of $\overline{C_6}$ (where $\overline{C_6}$ is the complement of C_6) in which the two triangles remain unsubdivided; in particular, $\overline{C_6}$ is a prism. A *wheel* is a graph that consists of a hole (called the *rim*) and an additional vertex (called the *center*) that has at least three neighbors in the rim. A *universal wheel* is a wheel whose center is adjacent to all the vertices of the rim, and is denoted by W_n if the rim is of length n. A *twin wheel* is a wheel whose center is adjacent to three consecutive vertices of the rim, and to no other vertices of the rim.

For a vertex v in a graph G, $N_G(v)$ (or simply $N(v)$ when clear from context) denotes the set of neighbors of v in G. $N[v] = N(v) \cup \{v\}$.

A *clique* in a graph G is a (possibly empty) set of pairwise adjacent vertices of G and a *stable set* in G is a (possibly empty) set of pairwise nonadjacent vertices of G. The *clique number* of G, denoted $\omega(G)$, is the maximum size of a clique in G and the *stability number* of G, denoted by $\alpha(G)$, is the maximum size of a stable set in G.

Two sets X and Y of vertices are said to be *complete to each other* if every vertex in X is adjacent to every vertex in Y; X and Y are *anticomplete* if no vertex of X is adjacent to any vertex of Y.

The complement of a graph G is denoted by $\overline{G}$. As usual, a *component* of G is a maximal connected induced subgraph of G. A graph is *anticonnected* if its complement is connected. An *anticomponent* of a graph G is a maximal anticonnected induced subgraph of G. (Thus, H is an anticomponent of G if and only if $\overline{H}$ is a component of $\overline{G}$.) Note that anticomponents of a graph G are pairwise complete to each other in G.

Let G be a graph, $S \subseteq V(G)$ and $F \subseteq E(G)$. We use $G \setminus S$ to denote G with the vertices of S deleted and $G \setminus F$ to denote G with the edges of F deleted. The subgraph of G induced by S is denoted by $G[S]$ (so $G \setminus S = G[V(G) \setminus S]$). We say that S is a *vertex cutset* of G if $G \setminus S$ is disconnected. A *clique cutset* of G is a vertex cutset that is a clique of G. Note that an empty set is a clique, and hence every disconnected graph has a clique cutset. A *star cutset* of G is a vertex cutset S that contains a vertex x that is complete to $S \setminus \{x\}$.

A *1-join* of a graph G is a partition (X_1, X_2) of its vertex set such that, for $i = 1, 2$, $|X_i| \geq 2$, X_i contains a nonempty subset A_i, that satisfy: A_1 is complete to A_2, and these are the only edges that go across the partition. A *2-join* of a graph G is a partition (X_1, X_2) of its vertex set such that, for $i = 1, 2$, $|X_i| \geq 3$, X_i contains nonempty disjoint subsets A_i and B_i that satisfy: A_1 is complete to A_2, B_1 is complete to B_2, and these are the only edges that go across the partition.

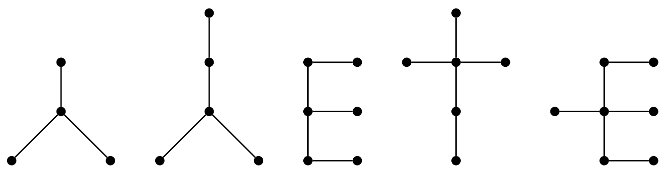

Figure 1: Claw Chair/Fork E Cross Trident

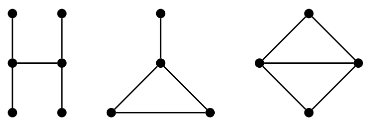

Figure 2: H Paw Diamond

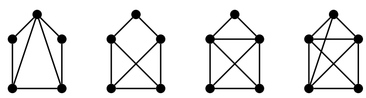

Figure 3: Gem $\overline{P_2 \cup P_3}$ HVN $K_5 - e$

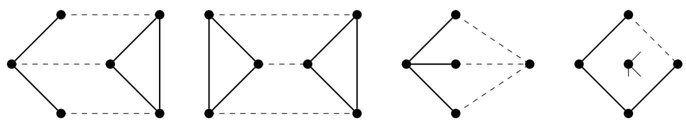

Figure 4: Pyramid Prism Theta Wheel
Dashed lines indicate paths

2 Classes χ-bounded by the function $f(x) = x + 1$

A hereditary class of graphs $\mathcal{G}$ is *χ-bounded* if there exists a function f such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq f(\omega(G))$. χ-Bounded classes were introduced by Gyárfás [26] in the 1980s as a generalization of perfect graphs. The bound $\chi(G) \leq \omega(G) + 1$ on the chromatic number χ is sometimes called the *Vizing bound* since Vizing proved that it holds for line graphs G of simple graphs [51].

By the following lemma, HC holds for all graph classes that satisfy the Vizing bound. In their survey on χ-bounds [46], Schiermeyer and Randerath have a section on graphs satisfying the Vizing bound (they have focused on classes defined by a finite number of excluded subgraphs). In this section we add to their list more graph classes for which it is known that the Vizing bound holds.

Lemma 2.1. *Let $\mathcal{G}$ be a hereditary class of graphs that is χ-bounded by function $f(x) = x+1$ (i.e. for every $G \in \mathcal{G}$, $\chi(G) \leq \omega(G)+1$). Then Hadwiger's Conjecture holds for $\mathcal{G}$.*

Proof. Let $G \in \mathcal{G}$ be a minimal counterexample. So G has no K_{t+1} minor, $\chi(G) = t + 1$ and every proper induced subgraph G' of G is t-colorable. Since G has no K_{t+1} minor, $\omega(G) \leq t$. If $\omega(G) < t$ then, since $G \in \mathcal{G}$, $\chi(G) \leq t$, a contradiction. So $\omega(G) = t$. Let K be a clique in G of size t. Since G is not t-colorable, $V(G) \setminus K \neq \emptyset$. Since G is not t-colorable and every proper induced subgraph of it is, G cannot have a clique cutset. It follows that $G \setminus K$ is connected (since otherwise K is a clique cutset) and every vertex of K has a neighbor in $V(G) \setminus K$ (since otherwise, if $k \in K$ has no neighbor in $V(G) \setminus K$, $K \setminus \{k\}$ is a clique cutset) and hence G has a K_{t+1} minor, a contradiction. □

The following classes of graphs satisfy the Vizing bound, and hence HC.

- perfect graphs

  A graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$. So by their definition perfect graphs satisfy HC and are χ-bounded by $f(x) = x$. By the Strong Perfect Graph Theorem [17] they are precisely the class of (odd hole, odd antihole)-free graphs (where an *antihole* is a complement of a hole).

- line graphs of (simple) graphs

  This class satisfies the Vizing bound by Vizing's Theorem [51]. The class is characterized by 9 forbidden induced subgraphs [4]. Coloring is NP-hard on line graphs (as edge-coloring is NP-hard).

- graphs in which no cycle has a unique chord (unichord-free graphs)

  In [50], a decomposition theorem is obtained for this class using vertex cutsets of size at most 2 and 1-joins, which is then used to obtain a polynomial-time coloring algorithm for this class, which shows that the class is χ-bounded by $f(x) = \max\{3, x\}$.

- (diamond, even hole)-free graphs

  In [33], a decomposition theorem is obtained for this class using bisimplicial cutsets (a certain type of vertex cutset whose vertices partition into 2 cliques) and 2-joins. This decomposition theorem is then used to show that every graph in this class has a vertex that is either of degree 2 or whose neighborhood is a clique, which implies that the class is χ-bounded by $f(x) = \max\{3, x\}$, and gives a polynomial-time coloring algorithm.

- (theta, wheel)-free graphs

  In [40], a decomposition theorem is obtained for this class using clique cutsets and 2-joins, which is then used to obtain a polynomial-time coloring algorithm in [41], which shows that the class is χ-bounded by $f(x) = \max\{3, x\}$.

- (claw, $K_5 \setminus e$)-free graphs [32] and more generally, (chair, $K_5 \setminus e$)-free graphs [43]

  Note that the claw and $K_5 \setminus e$ are two of the 9 graphs in the forbidden induced subgraph characterization of line graphs mentioned above.

- (chair, HVN)-free graphs [43]

- (triangle, C_5, trident)-free graphs [24]

  The theorem was first proved for a class of graphs in which there exists a shortest odd cycle such that all other vertices are with distance two of the cycle. It was then argued that a minimum counterexample belongs to this class. Randerath [43] has conjectured that (triangle, trident)-free graphs satisfy the Vizing bound.

- (paw, H)-free graphs, (paw, E)-free graphs, and (paw, cross)-free graphs [43]

  As pointed out in [46], Olariu [37] proved that connected paw-free graphs which contain a triangle are exactly the complete multipartite graphs with at least 3 parts. So for a set $\mathcal{B}$ of graphs, (triangle, $\mathcal{B}$)-free graphs satisfy the Vizing bound if and only if (paw, $\mathcal{B}$)-free graphs satisfy the Vizing bound.

- ($2K_2$, gem)-free graphs [9, 30]

  A ($2K_2$, gem)-free graph G is either perfect or contains a 5-hole; in the latter case, it is shown in [9] that it either has chromatic number equal to clique number or is obtained from the 5-hole by blowing vertices up into independent sets, and thus has chromatic number 3. A different proof is given in [30].

- ($2K_2$, $\overline{P_2 \cup P_3}$)-free graphs [30]

  This is proved using a structural result about the complementary class of graphs from [14].

- (P_6, E, diamond)-free graphs [29]

  A (P_6, E, diamond)-free graph is either perfect or contains a 5-hole; the result is obtained by studying the relationship of vertices to a particular 5-hole.

- class $\mathcal{G}_{\mathrm{U}}$, defined in Section 7

3 Graphs with stability number 2

HC has been studied for graphs with stability number 2 (an obvious reason why the chromatic number would be high), but remains open for this class. In [47], Seymour states: "if HC is true for graphs G with $\alpha(G) = 2$, then it is probably true in general."

Theorem 3.1. ([39]) *HC holds for all H-free graphs G with $\alpha(G) = 2$, where H is any graph on 4 vertices with $\alpha(H) = 2$.*

Theorem 3.2. ([34]) *HC holds for all H-free graphs G with $\alpha(G) = 2$, where H is any graph on 5 vertices with $\alpha(H) = 2$.*

Theorem 3.3. ([7]) *HC holds for all W_5-free graphs G with $\alpha(G) = 2$.*

Conjecture 3.4. ([39]) *Let G be a graph with n vertices and $\alpha(G) \leq 2$. Then G contains a clique minor of size $\frac{n}{2}$.*

In [39], it is shown that for graphs of independence number at most 2, HC is equivalent to Conjecture 3.4.

Theorem 3.5. ([20]) *Let G be a graph with n vertices and $\alpha(G) \leq 2$. Assume that some clique of G has cardinality at least $\frac{n}{4}$, and at least $\frac{n+3}{4}$ if n is odd. Then G has a clique minor of size $\lceil \frac{n}{2} \rceil$ (and so HC holds for G).*

4 Extensions of line graphs of simple graphs

Reed and Seymour [44] proved HC for line graphs of loopless multigraphs. Chudnovsky and Fradkin [15] extended this result by proving that HC holds for quasi-line graphs. A graph G is a *quasi-line graph* if for every $v \in V(G)$, $N(v)$ can be expressed as the union of two cliques.

Theorem 4.1. ([15]) *HC holds for quasi-line graphs.*

The proof uses the structure theorem for quasi-line graphs from [19], and the following result:

Theorem 4.2. ([18]) *If G is a quasi-line graph then $\chi(G) \leq \frac{3}{2}\omega(G)$.*

A *circular arc graph* is an intersection graph of arcs on a circle. A *proper circular arc graph* (or *circular interval graph*) is a circular arc graph where no arc properly contains another. Proper circular arc graphs are quasi-line. The structure theorem from [19] essentially states that quasi-line graphs can be decomposed by a non-crossing sequence of decompositions by homogeneous sets, homogeneous pairs and 2-joins (with all the special sets being cliques in all these cutsets) into proper circular arc graphs.

The fact that proper circular arc graphs satisfy HC was also proved independently in [5]. HC remains open for circular arc graphs. We remark that coloring is NP-hard for circular arc graphs [25], but polynomial for proper circular arc graphs [38]. Also in [28] it is shown that if G is a circular arc graph, then $\chi(G) \leq \frac{3}{2}\omega(G)$.

5 Applications of Theorem 4.1

Theorem 4.1 is used in [48] to obtain the following results.

Theorem 5.1. ([48]) *HC holds for graphs G that do not contain a hole of length between 4 and $2\alpha(G)$.*

Theorem 5.2. ([48]) *HC holds for graphs G with $\alpha(G) \geq 3$ which do not contain a hole of length between 4 and $2\alpha(G) - 1$.*

The following results use the fact that HC holds for proper circular arc graphs. A *trivial* anticomponent of a graph G is one that has just one vertex; a *nontrivial* anticomponent of G is one that has at least two vertices.

Theorem 5.3. *HC holds for* (C_4, C_5, P_7)*-free graphs.*

Proof. If a (C_4, C_5, P_7)-free graph does not contain a C_7, then it is perfect. It was shown in [11] that a (C_4, C_5, P_7)-free graph which contains a C_7 and does not admit a clique-cutset has at most one non-trivial anticomponent and this anticomponent is a proper circular arc graph. Thus the theorem follows from [15] or [5]. □

A *pan* is a graph consisting of a hole and an additional vertex which is adjacent to exactly one vertex of the hole. Note that a pan contains a claw, so claw-free graphs are pan-free.

Theorem 5.4. *HC holds for (pan, even hole)-free graphs.*

Proof. It was shown in [10] that a (pan, even hole)-free graph that does not admit a clique-cutset has at most one non-trivial anticomponent and this anticomponent is a unit circular arc graph (that is, a graph which can be represented as the intersection graph of unit length arcs of a circle), and thus a proper circular arc graph. Thus the theorem follows from [15] or [5]. □

6 Inflations of graphs

Given graphs G and H, we say that G is *obtained by blowing up each vertex of H to a clique* provided that there exists a partition $\{X_v\}_{v \in V(H)}$ of $V(G)$ into (possibly empty) cliques such that for all distinct $u, v \in V(H)$, if $uv \in E(H)$, then X_u is complete to X_v in G, and if $uv \notin E(H)$, then X_u is anticomplete to X_v in G. This is also referred to as substituting clique K_u for vertex u (for all u). G is also referred to as an *inflation* of H.

Theorem 6.1. ([13]) *HC holds for any inflation of any 3-colorable graph.*

So by the above theorem HC holds for example for inflations of outerplanar graphs. More recently the following classes have been shown to be 3-colorable (and hence by Theorem 6.1 HC holds for their inflations):

- (triangle, theta)-free graphs

  In [42], a decomposition theorem is obtained for this class using star cutsets, which is then used to show that this class is 3-colorable (and in fact colorable in polynomial time).

- (triangle, $IS\,K_4$)-free graphs

  In [16], this class is shown to be 3-colorable (and in fact colorable in polynomial time), by using decomposition by clique cutsets into complete bipartite graphs and ($IS\,K_4$, triangle, $K_{3,3}$)-free graphs, and then using star cutset decompositions to prove that every ($IS\,K_4$, triangle, $K_{3,3}$)-free graph has a vertex of degree at most 2.

The following conjecture was proposed in [35] (which, if true, would imply HC for $IS\,K_4$-free graphs).

Conjecture 6.2. *Every $IS\,K_4$-free graph is 4-colorable.*

We now give another consequence of Theorem 6.1. We *sign* a graph by assigning $0, 1$ weights to its edges. A graph is *odd-signable* if there exists a signing that makes the sum of the weights in every chordless cycle (including triangles) odd. Even-hole-free graphs are clearly odd-signable: assign weight 1 to each edge. Odd-signable graphs are precisely the class of (theta, prism, even wheel)-free graphs (where an *even wheel* is a wheel whose center has an even number of neighbors on the rim) [21]. A *cap* is a graph that consists of a hole H and a vertex x that has exactly two neighbors in H, that are furthermore adjacent. The structure of (cap, even hole)-free graphs, and more generally (cap, 4-hole)-free odd-signable graphs has been studied in [21, 22, 12], where the following results are obtained.

Graph G is obtained from graph H by *adding a universal clique* if G consists of H together with (a possibly empty) clique K, and all edges between vertices of K and vertices of H.

Theorem 6.3. ([12]) *Let G be a (cap, 4-hole)-free graph that contains a hole and has no clique cutset. Let F be any maximal induced subgraph of G with at least 3 vertices that is triangle-free and has no clique cutset. Then G is obtained from F by first blowing up vertices of F into nonempty cliques, and then adding a universal clique. Furthermore, any graph obtained by this sequence of operations starting from a (triangle, 4-hole)-free graph with at least 3 vertices and no clique cutset is (cap, 4-hole)-free and has no clique cutset.*

Theorem 6.4. ([22]) *Every (triangle, 4-hole)-free odd-signable graph has a vertex of degree at most 2.*

HC for (cap, 4-hole)-free odd-signable graphs follows from Theorems 6.1, 6.3 and 6.4. We now give a direct proof of this to illustrate some ways graph structure can be used to prove HC.

Theorem 6.5. *HC holds for (cap, 4-hole)-free odd-signable graphs, and hence for (cap, even hole)-free graphs.*

Proof. Suppose not and let G be a minimal counterexample. So G must be anticonnected, cannot have a clique cutset and must contain a hole (since graphs without holes are perfect). Let F be any maximal induced subgraph of G with at least 3 vertices that is triangle-free and has no clique cutset. Then by Theorem 6.3, G is obtained from F by blowing up vertices of F into cliques. By Theorem 6.4, F contains a vertex of degree 2. Let x_1, x_2, x_3 be a path in F such that x_2 is of degree 2 in F. Let X_1, X_2, X_3 be the corresponding cliques in G. So X_2 is complete to $X_1 \cup X_3$, and X_1 is anticomplete to X_3.

Let $k = \chi(G)$, let x be a vertex in X_2, let $X_2' = X_2 \setminus \{x\}$ and let $G' = G \setminus x$. Since G is a minimal counterexample, $\chi(G') = k - 1$. Let c be a proper $(k-1)$-coloring of G', using colors $1, \dots k-1$. For any $A \subseteq V(G')$, let $c(A)$ be the set of colors used on A. Since $\chi(G) = k$, $|c(X_1 \cup X_2' \cup X_3)| = k - 1$. W.l.o.g. $|X_1| \leq |X_3|$. Let X_1' be the set of vertices of X_1 that are colored with a color not used on X_3, and let $X_1'' = X_1 \setminus X_1'$. Let X_3' be the set of vertices of X_3 that are colored with a color not used on X_1, and let $X_3'' = X_1 \setminus X_3'$. It follows that $|X_1'| \leq |X_3'|$ and $|c(X_1' \cup X_2' \cup X_3)| = k - 1$. Now $X_1' \neq \emptyset$, since otherwise $X_2 \cup X_3$ is a clique of size k. Let $X_1' = \{a_1, \dots, a_t\}$ and $X_3' = \{b_1, \dots, b_{t'}\}$. So $t \leq t'$. Let $i \in \{1, \dots, t\}$ and let C_i be the connected component of the subgraph of G' induced by the vertices colored with colors from $c(\{a_i, b_i\})$ that contains a_i. If C_i does not contain b_i then by flipping colors on C_i we obtain a coloring of G' in which color $c(\{a_i\})$ does not appear in $N_G(x)$ and hence that color can be used on x to obtain a $(k-1)$-coloring of G, a contradiction. So C_i contains b_i, and hence there exists an a_i, b_i-path P_i in C_i. Let $P_i' = P_i \setminus \{b_i\}$ and let b_i' be the neighbor of b_i in P_i. Note that b_i' is complete to X_3. Furthermore, the paths $P_1', \dots, P_t'$ are vertex disjoint. It follows that the subgraph of G induced by $X_2 \cup X_3$ together with the paths $P_1', \dots, P_t'$ contains a K_k minor, a contradiction. □

7 Universal and twin wheel only graphs

A *three-path-configuration* (or *3PC* for short) is any theta, pyramid, or prism. A *proper wheel* is a wheel that is neither a universal wheel nor a twin wheel. A *Truemper configuration* is any 3PC or wheel. In [8], the following classes of graphs are studied:

- $\mathcal{G}_{\mathrm{UT}}$ =(3PC, proper wheel)-free graphs
  (so the only Truemper configurations that graphs in $\mathcal{G}_{\mathrm{UT}}$ may contain are universal wheels and twin wheels);

- $\mathcal{G}_{\mathrm{U}}$ = (3PC, proper wheel, twin wheel)-free graphs
  (so the only Truemper configurations that graphs in $\mathcal{G}_{\mathrm{U}}$ may contain are universal wheels);

- $\mathcal{G}_{\mathrm{T}}$ = (3PC, proper wheel, universal wheel)-free graphs
  (so the only Truemper configurations that graphs in $\mathcal{G}_{\mathrm{T}}$ may contain are twin wheels).

A *hyperhole* is a graph H whose vertex set can be partitioned into $k \geq 4$ nonempty cliques, call them $X_1, \dots, X_k$ (with subscripts understood to be in $\mathbb{Z}_k$), such that for all $i \in \mathbb{Z}_k$, X_i is complete to $X_{i-1} \cup X_{i+1}$ and anticomplete to $V(H) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$; under these circumstances, we say that the hyperhole H is of *length* k, and we also write that "$H = X_1, \dots, X_k, X_1$ is a hyperhole"; furthermore, we say that $(X_1, \dots, X_k)$ is a *good partition* of the hyperhole H. A *k-hyperhole* is a hyperhole of length k, and a *long hyperhole* is a hyperhole of length at least five.

A *hyperantihole* is a graph A whose vertex set can be partitioned into $k \geq 4$ nonempty cliques, call them $X_1, \dots, X_k$ (with subscripts understood to be in $\mathbb{Z}_k$), such that for all $i \in \mathbb{Z}_k$, X_i is anticomplete to $X_{i-1} \cup X_{i+1}$ and complete to $V(A) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$; under these circumstances, we say that the hyperantihole A is of *length* k, and we also write that "$A = X_1, \dots, X_k, X_1$ is a hyperantihole"; furthermore, we say that $(X_1, \dots, X_k)$ is a *good partition* of the hyperantihole A. A *k-hyperantihole* is a hyperantihole of length k, and a *long hyperantihole* is a hyperantihole of length at least five. Note that the complement of a hyperantihole need not be a hyperhole.

A *ring* is a graph R whose vertex set can be partitioned into $k \geq 4$ nonempty sets, say $X_1, \dots, X_k$ (with subscripts understood to be in $\mathbb{Z}_k$), such that for all $i \in \mathbb{Z}_k$, X_i can be ordered as $X_i = \{u_1^i, \dots, u_{|X_i|}^i\}$ so that $X_i \subseteq N_R[u_{|X_i|}^i] \subseteq \dots \subseteq N_R[u_1^i] = X_{i-1} \cup X_i \cup X_{i+1}$. Under these circumstances, we say that the ring R is of *length* k, as well as that R is a *k-ring*. A ring is *long* if it is of length at least five. Furthermore, we say that $(X_1, \dots, X_k)$ is a *good partition* of the ring R. We observe that every k-hyperhole is a k-ring.

Given a graph G and distinct vertices $u, v \in V(G)$, we say that *u dominates v* in G, or that v is *dominated* by u in G, provided that $N_G[v] \subseteq N_G[u]$. The following lemma follows directly from the definitions.

Lemma 7.1. *Let G be a graph, and let $(X_1, \dots, X_k)$, with $k \geq 4$ and subscripts understood to be in $\mathbb{Z}_k$, be a partition of $V(G)$. Then G is a k-ring with good partition $(X_1, \dots, X_k)$ if and only if all the following hold:*

(a) $X_1, \dots, X_k$ are cliques;

(b) for all $i \in \mathbb{Z}_k$, X_i is anticomplete to $V(G) \setminus (X_{i-1} \cup X_i \cup X_{i+1})$;

(c) *for all* $i \in \mathbb{Z}_k$, *some vertex of* X_i *is complete to* $X_{i-1} \cup X_{i+1}$*;*

(d) *for all* $i \in \mathbb{Z}_k$, *and all distinct* $y_i, y_i' \in X_i$, *one of* y_i, y_i' *dominates the other.*

Theorem 7.2. ([8]) *Every graph in* $\mathcal{G}_{UT}$ *either has a clique cutset or satisfies one of the following:*

- *G has exactly one nontrivial anticomponent, and this anticomponent is a long ring;*
- *G is (long hole,* $K_{2,3}$, $\overline{C_6}$*)-free;*
- $\alpha(G) = 2$, *and every anticomponent of G is either a 5-hyperhole or a (long hole,* $K_{2,3}$, $\overline{C_6}$*)-free graph.*

Theorem 7.3. ([8]) *Every graph in* $\mathcal{G}_U$ *either has a clique cutset or satisfies one of the following:*

- *G has exactly one nontrivial anticomponent, and this anticomponent is a long hole;*
- *all nontrivial anticomponents of G are isomorphic to* $\overline{K_2}$*.*

Theorem 7.4. ([8]) *Every graph in* $\mathcal{G}_T$ *either has a clique cutset or satisfies one of the following:*

- *G is a complete graph;*
- *G is a ring;*
- *G is a 7-hyperantihole.*

Theorem 7.3 is used to show that $\mathcal{G}_{\mathrm{U}}$ is χ-bounded by function $f(x) = x + 1$, and hence by Lemma 2.1, HC holds for $\mathcal{G}_{\mathrm{U}}$. In [36], HC is proved for $\mathcal{G}_{\mathrm{T}}$. We reproduce here some of the steps of this proof to highlight different proof techniques used to obtain this result (i.e. by Theorem 7.4 the graphs in $\mathcal{G}_{\mathrm{T}}$ are decomposed into basic classes that each require a different type of a proof that HC holds for them).

Theorem 7.5. ([36]) *For a ring R,*
$\chi(R) = \max\{\chi(H) :$ *H is a hyperhole contained in R*$\}$.
Furthermore, rings can be colored in polynomial time.

Hyperholes are quasi-line and so HC holds for hyperholes by Theorem 4.1. Hyperholes can also be seen as inflations of holes, and HC holds for them also by Theorem 6.1. In [36], the following direct proof of this was given (see also [48] for a proof of a slightly stronger result).

Lemma 7.6. (Lemma 7.1 in [36]) *HC holds for hyperholes.*

Proof. Let G be a hyperhole with good partition $(X_1, \dots, X_k)$. W.l.o.g. X_1 is the smallest of the sets $X_1, \dots, X_k$. Let $G' = G \setminus X_1$. Clearly $\chi(G) \leq \chi(G') + |X_1| = \omega(G') + |X_1|$. Let $j \in \{2, \dots, k-1\}$ be such that $|X_j \cup X_{j+1}| = \omega(G')$. By the choice of X_1, there exist $|X_1|$ vertex disjoint paths that each contain exactly one vertex from each of X_i, for $i \in \{1, \dots, k\} \setminus \{j, j+1\}$. But then the subgraph of G induced by these paths and $X_j \cup X_{j+1}$ contains a $K_{\chi(G)}$ minor. □

Lemma 7.7. (Lemma 7.2 in [36]) *HC holds for rings.*

Proof. Follows from Lemma 7.6 and Theorem 7.5. □

Lemma 7.8. (Lemma 7.3 in [36]) *HC holds for hyperantiholes.*

In [36], the above lemma is proved directly. Here is its proof using Theorem 3.5:

Let $G = (X_1, \dots, X_k)$ be a hyperantihole. Then clearly $\alpha(G) = 2$ and $k \geq 4$. By Theorem 3.5 it suffices to show that G has a clique of cardinality at least $\frac{n}{4}$, and at least $\frac{n+3}{4}$ if n is odd. We may assume that $n \geq 6$ since otherwise this is clearly true. For every $i = 1, \dots, k$, $V(G) \setminus X_i$ partitions into 2 cliques. W.l.o.g. X_1 is the smallest of the sets $X_1, \dots, X_k$. Then $|X_1| \leq \frac{n}{k}$ and hence $|V(G) \setminus X_1| \geq n - \frac{n}{k} = \frac{k-1}{k}n \geq \frac{3}{4}n$ (as $k \geq 4$). It follows that $G \setminus X_1$ has a clique of size $\frac{3}{8}n$ which is at least $\frac{n+3}{4}$ (as $n \geq 6$). □

Here is another proof, using Theorems 3.2 and 3.1: If $k \neq 5$ then G is C_5-free and hence HC holds for G by Theorem 3.2. If $k = 5$ then G is C_4-free and hence HC holds for G by Theorem 3.1. □

Theorem 7.9. (Theorem 7.4 in [36]) *HC holds for $\mathcal{G}_T$.*

Proof. Follows from Theorem 7.4 and Lemmas 7.7 and 7.8. □

Clearly every minimum counterexample to HC has no clique cutset and is anticonnected. If G satisfies the first bullet of Theorem 7.2, then HC holds for G by Theorem 7.7. If G satisfies the third bullet of Theorem 7.2, then G is C_4-free or C_5-free and hence HC holds for G by Theorem 3.1 or Theorem 3.2. So the following conjecture implies HC for $\mathcal{G}_{\mathrm{UT}}$.

Conjecture 7.10. *If G is (long hole, $K_{2,3}$, $\overline{C_6}$)-free and $\alpha(G) \geq 3$ then HC holds for G.*

The following property of (long hole, $K_{2,3}$)-free graphs may be of use for proving the above conjecture.

Theorem 7.11. *If G is (long hole, $K_{2,3}$)-free then for some vertex v in G, $\alpha(G[N(v)]) \leq 2$.*

Proof. In [1], it is shown that every graph that is (theta, pyramid, 1-wheel)-free has a vertex v such that $\alpha(G[N(v)]) \leq 2$ (where a *1-wheel* is a wheel whose rim contains a subpath a, b, c such that the center is adjacent to b and anticomplete to $\{a, c\}$). Since (long hole, $K_{2,3}$)-free graphs are a subclass of (theta, pyramid, 1-wheel)-free graphs, the result follows. □

References

[1] P. Aboulker, P. Charbit, N. Trotignon and K. Vušković. Vertex elimination orderings for hereditary graph classes. *Discrete Mathematics* 338 (2015) 825-834.

[2] K. Appel and A. Haken. Every planar map is 4-colorable. Part I. Discharging. *Illinois J. Math.* 21 (1977) 429-490.

[3] K. Appel, A. Haken and J. Koch. Every planar map is 4-colorable. Part II. Reducibility. *Illinois J. Math.* 21 (1977) 491-567.

[4] L.W. Beineke. Characterization of derived graphs. *Journal of Combinatorial Theory* 9 (1970) 129-135.

[5] N. Belkale and S. L. Chandran. Hadwiger's conjecture for proper circular arc graphs. *European J. Combin.* 30(4) (2009) 946-956.

[6] Z. Blázsik, M. Hujter, A. Pluhár and Zs. Tuza. Graphs with no induced C_4. *Discrete Math.* 115 (1993) 51-55.

[7] C. Bosse. A note on Hadwiger's conjecture for W_5-free graphs with independence number 2. *Discrete Math.* 342(12) (2019) 5pp.

[8] V. Boncompagni, I. Penev and K. Vušković. Clique-cutsets beyond chordal graphs. *Journal of Graph Theory* 91(2) (2019), 192-246.

[9] C. Brause, B. Randerath, I. Schiermeyer and E. Vumar. On the chromatic number of $2K_2$-free graphs. *Discrete Appl. Math.* 253 (2019) 14-24.

[10] K. Cameron, S. Chaplick and C. T. Hoàng. On the structure of pan, even hole-free graphs. *Journal of Graph Theory* 87(1) (2018) 108-129.

[11] K. Cameron, S. Huang, I. Penev and V. Sivaraman. The class of (P_7, C_4, C_5)-free graphs: decomposition, algorithms, and χ-boundedness. *Journal of Graph Theory* 93(4) (2020), 503-552.

[12] K. Cameron, S. Huang, M.V.G. da Silva and K. Vušković. Structure and algorithms for (cap, even hole)-free graphs. *Discrete Mathematics* 341(2) (2018) 463-473.

[13] C. J. Casselgren and A. S. Pedersen. Hadwiger's conjecture for inflations of 3-chromatic graphs. *European J. Combin.* 51 (2016) 99-108.

[14] S.A. Choudum and T. Karthick Maximal cliques in $\{P_2 \cup P_3, C_4\}$-free graphs *Discrete Math.* 310 (2010) 3398-3403.

[15] M. Chudnovsky and A.O. Fradkin. Hadwiger's conjecture for quasi-line graphs. *Journal of Graph Theory* 59 (2008) 17-33.

[16] M. Chudnovsky, C.-H. Liu, O. Schhaudt, S. Spirkl, N. Trotignon and K. Vušković. Triangle-free graphs that do not contain an induced subdivision of K_4 are 3-colorable. *Journal of Graph Theory* 92(2) (2019) 67-95.

[17] M. Chudnovsky, N. Robertson, P. Seymour, R. Thomas. The strong perfect graph theorem. *Annals of Math. 164* (2006) 51-229.

[18] M. Chudnovsky and A. Ovetsky. Coloring quasi-line graphs. *Journal of Graph Theory* 54 (2007) 41-50.

[19] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. *Surveys in Combinatorics*, 153-171, *London Math. Soc. Lecture Note Ser.* 327, Cambridge Univ. Press, Cambridge, 2005.

[20] M. Chudnovsky and P. Seymour. Packing seagulls. *Combinatorica* 32 (2012) 251-282.

[21] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković. Even and odd holes in cap-free graphs. *Journal of Graph Theory* 30 (1999) 289-308.

[22] M. Conforti, G. Cornuéjols, A. Kapoor and K. Vušković. Triangle-free graphs that are signable without even holes. *Journal of Graph Theory* 34 (2000) 204-220.

[23] G.A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Mat. Soc.* 27 (1952) 85-92.

[24] G. Fan, B. Xu, T. Ye and X. Yu. Forbidden subgraphs and 3-colorings. *SIAM J. Discrete Math.* 28 (2017) 1226-1256.

[25] M.R. Garey, D.S. Johnson, G.L. Miller and C.H. Papadimitriou. The complexity of coloring circular arcs and chords. *SIAM J. Alg. Disc. Meth.* 1 (1980) 216-227.

[26] A. Gyárfás, Problems from the world surrounding perfect graphs. *Zastosowania Matematyki* 19(3-4) (1987) 413-441.

[27] H. Hadwiger. Über eine Klassifikation der Streckenkomplexe. *Vierteljahrsschrift der Naturf. Gesellschaft in Zürich* 88 (1943) 133-142.

[28] I.A. Karapetyan. Coloring of arc graphs. *Akad. Nauk Armyan. SSR Dokl.* 70 (1980) 306-311 (in Russian).

[29] T. Karthick and F. Maffray. Vizing bound for the chromatic number of some graph classes. *Graphs Combin.* 32 (2016) 1447-1460.

[30] T. Karthick and S. Mishra. Chromatic bounds for some classes of $2K_2$-free graphs. *Discrete Math.* 341(11) (2018) 3079-3088.

[31] Ken-ichi Kawarabayashi. Hadwiger's conjecture. *Topics in Chromatic Graph Theory*, 73-93, Encyclopedia Math. Appl., 156, Cambridge Univ. Press, Cambridge, 2015.

[32] H. Kierstead. On the chromatic index of multigraphs without large triangles. *Journal of Combinatorial Theory B* 36 (1984) 156-160.

[33] T. Kloks, H. Müller and K. Vušković. Even-hole-free graphs that do not contain diamonds: a structure theorem and its consequences. *Journal of Combinatorial Theory B* 99 (2009) 733-800.

[34] M. Kriesell. On Seymour's strengthening of Hadwiger's conjecture for graphs with certain forbidden subgraphs. *Discrete Mathematics* 310 (2010) 2714-2724.

[35] Lévêque, Benjamin, Frédéric Maffray, and Nicolas Trotignon. On graphs with no induced subdivision of K_4. *Journal of Combinatorial Theory B* 102(4) (2012) 924-947.

[36] F. Maffray, I. Penev and K. Vušković. Coloring rings. Submitted. ArXiv:1907.11905v1

[37] Stephan Olariu. Paw-free graphs. *Inform. Process. Lett.* 28(1) (1988) 53-54.

[38] J.B. Orlin, M.A. Bonuccelli and D.P. Bovet. An $O(n^2)$ algorithm for coloring proper circular arc graphs. *SIAM J. Alg. Disc. Meth.* 2(2) (1981) 88-93.

[39] M.D. Plummer, M. Stiebitz and B. Toft. On a special case of Hadwiger's Conjecture. *Discuss. Math. Graph Theory* 23 (2003) 333-363.

[40] M. Radovanović, N. Trotignon and K. Vušković. The (theta, wheel)-free graphs, Part II: structure theorem. *Journal of Combinatorial Theory B* 143 (2020) 148-184.

[41] M. Radovanović, N. Trotignon and K. Vušković. The (theta, wheel)-free graphs, Part III: cliques, stable sets and coloring. *Journal of Combinatorial Theory B* 143 (2020) 185-218.

[42] M. Radovanović and K. Vušković. A class of 3-colorable triangle-free graphs. *Journal of Graph Theory* 72(4) (2013) 430-439.

[43] B. Randerath. The Vizing bound for the chromatic number based on forbidden pairs. Ph.D. thesis. RWTH Aachen, Shaker Verlag (1998).

[44] B. Reed and P. Seymour. Hadwiger's conjecture for line graphs. *European Journal of Combinatorics* 25(6) (2004) 873-876.

[45] N. Robertson, P. Seymour and R. Thomas. Hadwiger's conjecture for K_6-free graphs. *Combinatorica* 13 (1993) 279-361.

[46] I. Schiermeyer and B. Randerath. Polynomial χ-binding functions and forbidden induced subgraphs: a survey. *Graphs and Combinatorics* 35 (2019) 1-31.

[47] P. Seymour. Hadwiger's conjecture. *Open Problems in Mathematics*, 417-437, Springer, [Cham], 2016.

[48] Z.-X. Song and B. Thomas. Hadwiger's conjecture for graphs with forbidden holes. *SIAM Journal on Discrete Mathematics* 31(3) (2017) 1572-1580 .

[49] B. Toft. A survey of Hadwiger's conjecture. *Congressus Numerantium* 115 (1996) 249-283.

[50] N. Trotignon and K. Vušković. A structure theorem for graphs with no cycle with a unique chord and its consequences. *Journal of Graph Theory* 63(1) (2010) 31-67.

[51] V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz.* 3 (1937) 25-30.

[52] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.* 114 (1937) 570-590.