The Logic in Computer Science Column

BY

Yuri Gurevich

Computer Science and Engineering
University of Michigan, Ann Arbor, MI 48109, USA
gurevich@umich.edu

FEASIBILITY, SORITES AND VAGUENESS

Rohit Parikh
City University of New York
CS, Math, Philosophy
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Abstract

We look at the history of research on vagueness and the Sorites paradox. That search has been largely unsuccessful and the existing solutions are not quite adequate. But following Wittgenstein we show that the notion of a *successful language game* works. Language games involving words like "small" or "red" can be successful and people can use these words to cooperate with others. And yet, ultimately these words do not *have* a meaning in the sense of a tight semantics. It is just that most of the time these games work. It is fine to say, "the light is green and we can go," even though the color word "green" does not actually *have* a semantics.

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According to Hindu scriptures, the evil king Hiranyakashyapu prayed to the god *Brahma* to grant him the following boon:

> *Grant me that I not die within any residence or outside any residence, during the daytime or at night, nor on the ground or in the sky. Grant me that my death not be brought by any being other than those created by you, nor by any weapon, nor by any human being or animal.*

Eventually, Hiranyakashyapu was killed by a creature (*Narasimha*) which was half man and half lion, not killed in day or night but at dusk, and not indoors nor outdoors but at the doorstep.

Evidently, Hiranyakashyapu did not realize that with ambiguous A, B, a conjunction $\neg A \wedge \neg B$ could be true even though $A \vee B$ is also true. He was not killed indoors and he was not killed outdoors even though he was killed "either indoor or outdoor". We will return to this point which is also discussed by Michael Dummett[1] as well as by Kit Fine.

[1] "But, now, consider a vague statement, for instance 'That is orange'. If the object pointed to is definitely orange, then of course the statement will be definitely true; if it is definitely some other colour, then the statement will be definitely false; but the object may be a borderline case, and then the statement will be neither definitely

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1 Background

The Megarian philosopher Eubulides (4th century BC) is usually credited with the first formulation of the following puzzle.

1. 1 grain of wheat does not make a heap.
2. if n grains do not make a heap then n+1 grains do not make a heap. Therefore,
3. 1 million grains don't make a heap.

This inductive argument can be replaced by a large number of applications of *modus ponens*

1. 1 grain of wheat does not make a heap.
2. If 1 grain doesn't make a heap, then 2 grains don't.
3. If 2 grains don't make a heap, then 3 grains don't.
   ...
4. If 999,999 grains don't make a heap, then 1 million grains don't. Therefore,
5. 1 million grains don't make a heap.

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true nor definitely false. But, in this instance at least, it is clear that, if a borderline case, the object will have to be on the borderline between being orange and being some other particular colour, say red. The statement 'That is red' will then likewise be neither definitely true nor definitely false: but, since the object is on the borderline between being orange and being red - there is no other colour which is a candidate for being the colour of the object - *the disjunctive statement, 'That is either orange or red' will be definitely true even though neither of its disjuncts is.*" (emphasis mine). (Dummett 1975)

Since *soros* is the Greek word for a heap, this puzzle is often known as the *Sorites paradox*. Here 2-4 replaced the inductive step, *if n grains do not make a heap then n+1 grains do not make a heap.*

There is a similar Indian story about a woman who cured her husband of his opium addiction. She weighed his usual opium ration against a ball of woolen thread. Then every day, she cut off about an inch or so of the ball and she weighed his opium ration against the reduced ball. The husband did not notice any difference from one day to the next, but over time, the ball became empty and the husband was cured.

This story must be recent as opium was only introduced into India by the Mughals in the 17th century, long after Eubulides.

1.1 Precisification and super-truth

One way to deal with this problem is epistemic. Namely that there is an n for which n grains do not make a heap but that $n + 1$ does but we do not know which n it is. We will say that a statement is super-true if it is true regardless of which n it is and it is super-false regardless of which n it is. In that case $(\exists n)(\neg H(n) \wedge H(n + 1))$ is super-true although we are unable to give an explicit n.

Kit Fine (Fine 75) says,

“In this section we shall argue for the super-truth theory, that a vague sentence is true if it is true for all admissible and complete specifications. An intensional version of the theory is that a sentence is true if it is true for all ways of making it completely precise ... As such, it is a sort of principle of non-pedantry : truth is secured if it does not turn upon what one means. Absence of meaning makes for absence of truth-value only if presence of meaning could make for diversity of truth-value. The theory is a partial vindication of the classical position. For the truth-conditions are, if not classical, then classical at a remove. There is but one rule linking truth to classical truth, viz. that truth is truth in each of a set of interpretations."

Fine manages to save a great deal of classical logic and our intuitions this way. Suppose there is doubt whether a certain patch is pink or red. Then, according to him, it is not (super)true that it is pink, it is not true that it is red, but it is true that it is either pink or red. For regardless of where we draw the boundary, it will be one or the other. And it is false that it is both pink and red for no path to a complete specification will render it both.

1.2 Fuzzy logic

Lotfi Zadeh addresses this problem by resorting to truth values properly between 0 and 1.

“More often than not, the classes of objects encountered in the real physical world do not have precisely defined criteria of membership. For example, the class of animals clearly includes dogs, horses, birds, etc. as its members, and clearly excludes such objects as rocks, fluids, plants, etc. However, such objects as starfish, bacteria, etc. have an ambiguous status with respect to the class of animals. The same kind of ambiguity arises in the case of a nmnber such as 10 in relation to the "class" of all real numbers which are much greater than 1."

“Clearly, the "class of all real numbers which are much greater than 1," or "the class of beautiful women," or "the class of tall men," do not constitute classes or sets in the usual mathematical sense of these terms. Yet, the fact remains that such imprecisely defined "classes" play an important role in human thinking, particularly in the domains of pattern recognition, communication of information, and abstraction." (Zadeh, 1965)

Zadeh addresses his concerns by allowing fuzzy truth values in the real interval [0,1]. Let $R(x)$ indicate that the object x is red. If the fuzzy truth value of $R(x)$ is close to zero then it means that x is pretty much not red. If it is close to 1 then x is pretty close to being red. Now if you have a series of objects $o_1, o_2, ..., o_{100}$ and the R values gradually go up from 0 to 1 then one could say that the objects are gradually becoming more red, although you might not be able to distinguish say o_{49} and o_{50}. Thus the Sorites paradox is defanged so to say.

But two difficulties arise.

One is that if $R(50)$ has value .5, then $\neg R(50)$ has value $1 - 0.5$ and so also 0.5. But then by Zadeh’s (original) formula for the truth value of the disjunction $R(50) \vee \neg R(50)$ has value max(.5,.5) = .5 which some might feel conflicts with the intuition that the disjunction is analytic and should have value[2] 1. Also the value of the implication $R(n) \rightarrow R(n+1)$ (that is $\neg R(n) \vee R(n+1)$) starts at 1, falls to .5 in the middle and then rises again to 1. Does this agree with *your* intuition?

Again, one would expect that even though fuzzy truth values lie between 0 and 1 they should at least be *interpersonal*. If I think that something is .3 red then you should also think that it is .3 red. But in a test I administered in Sicily, (Parikh 1991), I found that people gave very different fuzzy values to questions like “is a handkerchief an item of clothing?" or “is Sonia Gandhi an Indian?". A handkerchief is made of cotton but we do not usually wear it so to the first question, intuitions conflict. Similarly Sonia Gandhi was Italian by birth but became an Indian citizen so again intuitions conflicted. These conflicts were resolved by different people in different ways so there was no stable fuzzy value.[3] So how can fuzzy logic be used for *communication*? Are we living in a Tower of Babel?

But perhaps communication does not necessitate having the exact same truth value for a proposition and perhaps *some* agreement suffices and is useful. We will return to this question.

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[2]Kit Fine gives the right value here.

[3]to the question “Is Sonia Gandhi an Indian?" 6 people gave a low value, 9 people gave a middle value and 9 people gave a high value.

2 Logical issues

2.1 Strict finitism

”Strict finitism was first suggested as a conceivable position in the philosophy of mathematics by Bernays in his article ‘On Platonism in Mathematics’. It was argued for by Wittgenstein in *Remarks on the Foundations of Mathematics*; but, with his staunch belief that philosophy can only interpret the world, and has no business attempting to change it, he did not propose that mathematics be reconstructed along strict finitist lines - something which evidently calls for a far more radical overhaul of mathematical practice than does traditional constructivism. The only person, so far as I know, to declare his adherence to strict finitism and attempt such a reconstruction of mathematics is Esenin Volpin. But, even if no-one were disposed to accept the arguments in favour of the strict finitist position, it would remain one of the greatest interest, not least for the question whether constructivism, as traditionally understood, is a tenable position". (Dummett 1975).[4]

We might regard strict finitism as an extreme version of constructivism. But while definitely of interest, it has not caught on.

2.2 Almost consistent theories

Let us define an *inductive* set of natural numbers to be a nonempty set X such that if $n \in X$ then $n+1 \in X$. Clearly such a set will contain all large numbers and if it contains 0, it will contain all numbers.

Let us define a *bounded* set X to be a set such that $(\exists M)(\forall n)(n \in X \rightarrow n < M)$. Clearly a bounded set must be finite whereas an inductive set will be infinite. So can a set be both bounded and inductive? Seems not.

Yesenin Volpin points out that such sets do exist in some sense. For let H be the set of heartbeats in one's childhood. No one ceases to be a child in a single heartbeat. So if $n \in H$ then $n+1 \in H$. And yet, assuming at most a hundred heartbeats per minute, there are fewer than 10 million heartbeats before one reaches the age of eighteen. So H is bounded above by 10 million. H is both inductive and bounded.

But isn't it inconsistent to speak of H at all?

To make things easier let us replace 10 million by a mere 100. The inconsistency should be even more glaring.

Consider a set X of formulas $\{H(1), H(1) \rightarrow H(2), ...H(99) \rightarrow H(100), \neg H(100)\}$ This set is

[4]Note that there is some inconsistency in how Ysenin Volpin's name is spelled in English. Since we are quoting Dummett, we have left his spelling as it is. Elsewhere we will use ‘Yesenin Volpin’.

inconsistent as the conclusion $H(100)$ can be derived from the first 100 formulas in X contradicting the last formula. However, let T be some consistent theory, say T is PA = Peano Arithmetic. Let $T' = T \cup X$. Evidently T' is inconsistent since it includes X. However, let A be some formula of number theory (not involving the predicate H) whose proof in T' takes less than 100 lines. Then A is a theorem of PA.

Proof: Clearly the proof includes less than 100 formulas of the form $H(n) \rightarrow H(n+1)$ and hence some formula of the form $H(n) \rightarrow H(n+1)$ is absent. To fix thoughts suppose that the formula $H(50) \rightarrow H(51)$ is missing from the proof. Then extend the usual interpretation of PA by interpreting $H(k)$ as $k < 51$. All the formulas which *occur* in the proof become true and A becomes a theorem of PA. □

This means that even though, in the classical sense there is no set like H and the properties of the putative H are inconsistent, it still *works* to reason with H. Only a dogmatic person will insist that there are no children or no heartbeats in childhood "because the very notions are inconsistent".

2.3 Weak theories

Let T be a theory whose language includes that of number theory. Let R be a binary predicate expressible in the language of T. We will say that R is *decidable* with respect to T if for all numerals m, n either $T \vdash R(m,n)$ or $T \vdash \neg R(m,n)$. We will say that a unary function f is *provably recursive* with respect to T if there is a decidable R such that $T \vdash (\forall x)(\exists y)R(x,y)$ and moreover for all m, n, $T \vdash R(m,n) \leftrightarrow n = f(m)$.

Now if T is Peano Arithmetic then all primitive recursive functions (and more) are provably recursive relative to T. However, primitive recursive functions can create large terms like 10^{10} and even $10^{10^{10}}$. Are there more modest theories? It turns out that if we limit induction to bounded formulas[5], then exponentiation is no longer provably recursive in the resulting theory, called PB in (Parikh 1971) and $I\Delta_1^0$ later on. Thus we can live inside PB and be safe!!

2.4 Bounded logics

Imagine a system where some axioms are given as are some rules of inference. There is also a budget and a cost to each rule of inference. We could say that we accept a proof in such a system as long as it is within the budget. Moreover the language of the system consists of two parts $L_1 \cup L_2$. We could say that the system is *weakly sound* if every formula of L_1 which has an acceptable proof is true. It is presumed here that there is a notion of truth for L_1 but not for L_2. Then such a system could be useful in deriving facts expressible in L_1. Words like "small", "blue" belong to L_2 and they do not really have a "meaning". But they are useful. Just like

[5] A formula is bounded if all the quantifiers are bounded by terms obtained from the variables and 0 using only S, + and $\times$.

“blue" the word “green" does not have a semantics. But “the light is green and we can go" is a perfectly sound procedure. We should not confuse the soundness of that procedure with the existence of a semantics for “green".

3 Vagueness in real life

3.1 Vagueness and communication

In the following example, the usefulness of communication consists of a saving of time. Ann and Bob teach at the same college. Ann teaches Math and Bob teaches History. One day Ann telephones Bob from school.

Ann: Bob, can you bring my topology book in?

Bob: What does it look like?

Ann: It is blue.

Bob: OK.

Ann: Be sure to bring it, I am going to lunch now, but I need it for class at 2 PM.

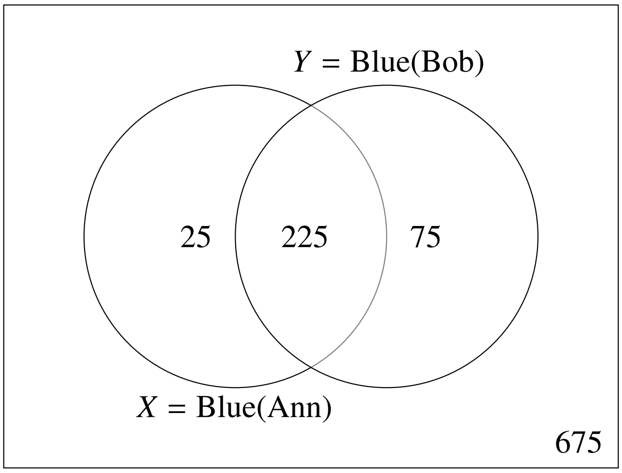
It so happens that Ann and Bob have somewhat different notions of what the word “blue" means, i.e. which things it applies to[6]

Now, among Ann’s 1,000 books, there are 250 that Ann would call blue whereas there are 300 that Bob would call blue.

Let X= Blue(Ann), the set of those books that Ann considers to be blue, and Y = Blue(Bob), the set of those books that Bob considers to be blue.

There are 225 books that both would call blue (i.e. they are in $X \cap Y$) and 675 that neither would (they are in $\overline{X} \cap \overline{Y}$). But there are 25 books that Ann, but not Bob would call blue ((they are in $X \cap \overline{Y}$)) and 75 books that Bob, but not Ann, would call blue (they are in $\overline{X} \cap Y$). So they would, if asked, disagree on 100 books.

[6]See Williamson in his *Vagueness* page 74. “In a borderline case, there will be variation between different speakers at the same time and in the same person at different times." Berlin and Kay 1991 make a similar point and I have verified it by experiment.



I shall assume that neither Ann nor Bob is aware of this. Now Ann intends Bob to look through the set X, but having his own notion of what blue is, he will look in Y = Blue(Bob). Here are the expected (average) number of books that Bob would look at in three cases.

If Bob had no information: 500 books on average. (If he is lucky, the first book that he looks at will be the topology book. If unlucky, the last book he looks at will be the topology book. The average is 500.5, or approximately 500.)

If Blue(Bob) were the same as Blue(Ann), i.e. as X: 125 books on average. (Same reasoning as before. Except that if Bob's notion of blue were exactly the same 250 books as Ann's, then he would confine his search to 250 books, so that he would look at 125 books on average.)

In the actual case, since the book is in X, with probability .9 it is also in Y. Since Bob actually looks in Y, and if necessary, in the complement of Y, with probability .9 he only needs to look through Y, or at most 300 books, yielding 150 average. With probability .1 he will not find it in Y. In time he will have looked through all of Y and will only need to look at about 350 further books in the complement of Y. Thus he will look at .9(150) + .1(300 + 350) = 200 books, which is the average in this case. (Note that he looks through $\overline{Y}$ since he has no idea about X). Thus Bob is saved considerable labour by what Ann said though his interpretation of "blue" is not what Ann intended. Instead of having to look through 500 he looks through 200. No *proposition* is conveyed by Ann to Bob for they do not share a semantics for *blue* but he is helped.[7]

So what happens to *the obligation to speak the truth*? Kant endorses truth as do both Moses and Buddha. But if there is no fact of the matter as to which books are indeed blue, then what happens to the obligation? Can we separate the obligation to speak the truth from the obligation to be helpful? Perhaps the latter obligation can be fulfilled without having the follow the former.

Here is Wittgenstein in his *Remarks on the Foundations of Mathematics*

[7]They do share a semantics in a weak sense in that when one is inclined to call an object blue, so is the other. But this is merely a correspondence in performance and not based on any reasoning. See also (Black 1937).

> "What we call counting is an important part of life's activities. Counting and calculating are not – e.g. – simply a pastime. Counting (and that means something like *this*) is a technique that's employed daily in the most various operations of our lives and that is why we learn to count as we do: with endless practice, with merciless exactitude; that is why it is in inexorably insisted that we shall all say two after one; three after two and so on. "But is this counting only a use? Isn't there also some truth corresponding to the sequence?" The truth is that counting has proved to pay – "then do you want to say that being true means being usable or useful?" No, not that but that it can't be said of the series of natural numbers anymore than of our language that it is true, but that is usable, and above all *it is used*".

Consider the following result due to Condorcet (his celebrated jury theorem). Suppose that there are n people who have to decide on the truth of some proposition P. Each of the people has a probability greater than $.5 + \epsilon$ of being right. Then with high probability the majority will be close to the truth and closer to the truth than any of the individuals. But it is presumed that the n people are independent and no one is influenced by another. So then the n people, expressing their own views, are *contributing* to finding the truth even though many of them are saying something false. Did they then discharge their obligation to speak the truth?

Coming back to Ann, we need not ask if Ann was speaking the truth when she said that the book was blue. There is indeed a 10% chance that Bob would disagree with her. But she did *help* Bob in his search for her book. I think Wittgenstein would like this example where a language game is successful even though it is not underpinned by a solid notion of objective truth.[8]

3.2 Vagueness and language games

Suppose that a community of people (like us) use words like "blue", "red", "large", "small" etc. and assign certain properties to the putative predicates. But on second thought it turns out that these properties are inconsistent and hence there *are* no such predicates (classically speaking). Does it follow that these people should constantly fall into confusion and be perpetually at war with each other? Not so.

Pragmatics normally depends on semantics. What it is useful to say depends on what is true. But the example with Bob and Ann shows that pragmatics can work and be beneficial without being underpinned by a semantics. If we say that something is blue, we are not really referring to a property "blueness", whether sharp or fuzzy. We are simply *saying something*, which can be treated (by others) as true under some circumstances and as false on others. But it is not as if their agreement or disagreement with us depends on a semantics that they and us *share*. Their utterances "it is blue" and ours have some correspondence. And that helps. But that correspondence does not depend on any semantics. Rather, their behavior and ours depends

[8] I would suggest that the reason this problem has been so thorny is that we have been looking for a semantics and a logic. We did not consider that there might be *successful* language games without there being a semantics to justify our language.

on some similarities in biology and some similarity in social experience. And that is good enough for most purposes. It is only when we put a lot of logical pressure on that similarity that problems like the Sorites paradox can arise.

For we saw that Bob and Ann can “communicate" even though they assign different extensions to the word “blue". We also saw that someone who believes that (i) 0 is small, that (ii) if n is small then so is $n + 1$ and that (iii) 10^{10} is not small, may succeed in making correct inferences provided only that she does not perform deductions of more than 10^{10} lines.

4 Dealing with non-transitivity

If we define $I(x, y)$ to mean that x, y are indiscriminable in some important way then I is reflexive and symmetric but may not not transitive. In other words, there can exist x, y, z such that x, y are indiscriminable, y, z are indiscriminable, but x, z are discriminable. Thus I not an equivalence relation. This fact is of course behind the Sorites paradox. This can create a problem in practical matters as when we are sorting socks after a wash and dry. Suppose we have six socks, A, B, C, D, E, F where the sets {A,B}, {C,D} and {E,F} are respectively from three different pairs of socks. Moreover each of A, B will match each of C, D. Each of C, D will match with each of E, F. However, because of intransitivity, A, B do not match E, F.

A ... B ... C ... D ... E ... F

Suppose now that all six socks have been washed and dried and, relying on matching, we pair together B,C which match. We also put together D, E which match.

A ... (B ... C) ... (D ... E) ... F

We are now left with A, F which do *not* match! How do we deal with this problem? We relied on indiscriminability which is not transitive. We could start over, but if there are a lot of socks we might be working for ever!

At first sight it looks as if finding a good matching might be an NP-complete problem, quite hard if there are a hundred socks.[9]

It turns out that there *is* a transitive relation J which depends on I, indiscriminability, but does not coincide with it. Given a sock s, let $M(s) = \{t : I(s, t)\}$ And let $J(s, t)$ mean that $M(s) = M(t)$. Then $J(s, t)$ implies $I(s, t)$ but is stronger. Moreover, J is transitive. Relying on I, we construct J and pair two socks s, t iff $J(s, t)$. This algorithm runs in n^2 time, showing that the original problem was not NP-complete.

To see that $J(x, y)$ implies $I(x, y)$ note the following. Suppose $J(x, y)$. Now $I(x, x)$ holds. Hence $x \in M(x)$. Given $J(x, y)$, $M(x) = M(y)$ and hence $x \in M(y)$. Ergo $I(x, y)$.

On the other hand I does not imply J, for in the example with A,..,F above, $I(B, C)$ holds. But

[9]No doubt this is one reason centipedes do not wear socks!

while $I(C, E)$ holds, $I(B, E)$ fails. Hence $E \in M(C)$ but $E \notin M(B)$. So $J(B, C)$ fails, and $I(B, C)$ fails to imply $J(B, C)$. (See Parikh et al 2001 for details).

5 Operational Semantics

Vaughan Pratt referred me to the Wikipedia article on this topic.
($https://en.wikipedia.org/wiki/Operational\_semantics$).
Here is a quote from that article.

> The concept of operational semantics was used for the first time in defining the semantics of Algol 68. The following statement is a quote from the revised ALGOL 68 report:
>
> “The meaning of a program in the strict language is explained in terms of a hypothetical computer which performs the set of actions which constitute the elaboration of that program" (Algol68, Section 2)
>
> The first use of the term "operational semantics" in its present meaning is attributed to Dana Scott (Plotkin 04). What follows is a quote from Scott’s seminal paper on formal semantics, in which he mentions the "operational" aspects of semantics.
>
> “It is all very well to aim for a more ‘abstract’ and a ‘cleaner’ approach to semantics, but if the plan is to be any good, the operational aspects cannot be completely ignored."

However, we do need a theory of what happens when *different agents* interpret the same vague predicate in different ways. Many cooks making incompatible decisions can spoil a broth, but they could also come up with a good feast. Vagueness is not always a disaster.

Consider the following situation. Country A moves some troops to its border with country B. If there is one soldier, we will not say, “A is massing troops," and if there are a million, we will. So “A is massing troops on the border with B" is a vague statement and interpreted by different governments and different generals in different ways. And yet we can predict *something*. We badly need a theory of how that happens.

6 Conclusion

We showed quite convincingly that vague predicates do not have a semantics and hence they do not have a logic. But they do have a *use* and we found how this use falls inside Wittgenstein’s requirement that a language game be useful and be used.

Here is a question I would raise – for the future. Suppose that people's reactions to "is it blue?" go according to experimental data. Then they will have different "semantics" for blue which will vary a little from person to person and from the same person to himself from time to time. But some algorithms will still "work". It is perfectly fine to say, "green means go and red means stop" even though both red and green are vague predicates. But Eubulides discovered that not everything works. So there does need to be a branch of analysis of algorithms which addresses this question. I.e. what works when different people working together on some task judge the same issue differently. We do use vague predicates in real life, and usually our applications work, more or less. But we lack a theory or *when* they will work.[10] We need such a theory. It is most likely to come from Computer Science or perhaps from computer scientists who are sympathetic to philosophy.

It should be possible to come up with a programming language in which "If Bob thinks it is blue then do A, and otherwise do B" can be used as an instruction in a program and one could evaluate the efficiency of such programs by rigorous means. That might well give us an insight into why we use vague predicates in daily life. If someone says "It is a good movie," we might not take it very seriously unless we know that our tastes are similar. If someone however says, "the sky was dark," we would surely infer *something*. Investigating such issues in detail is beyond the scope of the present paper But they are clearly relevant to the topic of social software (Parikh 2002).

To a question, "how many students passed the course?" there is likely to be an answer, reasonably accurate. But "how many of the students were tall?" will not have such a universally accepted answer. Luckily we do not *need* an answer here (unless we are putting together a basketball team). But what about "how much poverty is there in the world?" "Poverty" is a vague term. Someone who is starving would not be helped by the gift of a penny and Bill Gates would not become poor if we took away a dollar from him. So clearly there is room for a Sorites of poverty.

And yet poverty is one of the most important questions of our time.

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[10]"If we were to examine what everyman has in mind when each time he hears or repeats the word 'being,' we would gather most varied and most curious information. We would have to recognize that the most notorious state of the world today expresses itself in such inconspicuous fields as the range of meanings the word seems to have. In fact, that chaos may even have its roots there. But a still greater puzzle is that men nonetheless understand each other." Martin Heidegger, *What is Called Thinking?*,

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