Book Introduction by the Authors

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Essentials of Finitely Supported Structures

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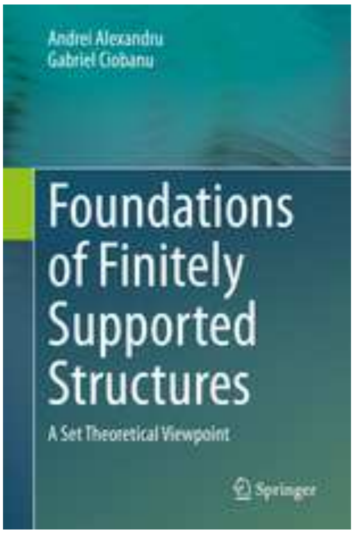
The aim here is to describe finitely supported sets and structures that were studied in the book "*Foundations of Finitely Supported Structures: a set theoretical viewpoint*" available at https://www.springer.com/gp/book/9783030529611.



This book provides a set theoretical study on the foundations of structures with finite supports. Finitely supported sets and structures are related to permutation models of Zermelo-Fraenkel set theory with atoms (ZFA) and to the theory of nominal sets. They were originally introduced in 1930s by Fraenkel, Lindenbaum and Mostowski to prove the independence of the axiom of choice and the other axioms in ZFA. Basic Fraenkel Model of ZFA was axiomatized and used by Gabbay and Pitts to study the binding, scope, freshness and renaming in programming languages and related formal systems. The axioms of this Fraenkel-Mostowski (FM) set theory are precisely the axioms of ZFA set theory together with an additional axiom for finite support. They are involved in the (hierarchical) construction of finitely supported sets; hereditary finitely supported sets are the sets constructed with respect to Fraenkel-Mostowski axioms over an infinite family of basic elements called atoms. An alternative approach that works in classical Zermelo-Fraenkel (ZF) set theory (i.e. without being necessary to consider an alternative set theory obtained by weakening the ZF axiom of extensionality) is related to the theory of nominal sets that are defined as usual ZF sets equipped with canonical permutation actions of the group of all one-to-one and onto transformations of a fixed infinite, countable ZF set formed by basic elements (i.e. by elements whose internal structure is not taken into consideration) satisfying a finite support requirement (traduced as 'for every element x in a nominal set there should exist a finite subset of basic elements S such that any one-to-one and onto transformation of basic elements that fixes S pointwise also leaves x invariant under the effect of the permutation action with who the nominal set is equipped').

Finitely supported sets are defined as finitely supported elements in the powerset of a nominal set. Using the categorical approach regarding nominal sets, one can prove that any function or relation that is defined from finitely supported functions and subsets using classical higher-order logic is itself finitely supported, with the requirement that we restrict any quantification over functions or subsets to range over ones that are finitely supported.

Inductively defined finitely supported sets involving the name-abstraction together with Cartesian product and disjoint union can encode a formal syntax modulo renaming of bound variables. In this way, the standard theory of algebraic data types can be extended to include signatures involving binding operators. In particular, there is an associated notion of structural recursion for defining syntax-manipulating functions and a notion of proof by structural induction. Certain generalizations of finitely supported sets are involved in the study of automata, programming languages or Turing machines over infinite alphabets; for this, a relaxed notion of finiteness called 'orbit finiteness' was defined; it means 'having a finite number of orbits (equivalence classes) under a certain group action'. Informally, the theory of finitely supported sets allows the computational study of structures which are possibly infinite, but contain enough symmetries such that they can be concisely represented and manipulated.

Finitely Supported Mathematics (shortly, FSM) is focusing on the foundations of set theory. In order to describe FSM as a theory of finitely supported algebraic structures (that are finitely supported sets together with finitely supported internal algebraic operation/laws or with finitely supported relations), we use nominal sets (ignoring the requirement that the set A of atoms is countable) which by now on will be called invariant sets motivated by Tarski's approach regarding logicality (i.e. a logical notion is defined by Tarski as one that is invariant under the permutations of the universe of discourse).

There is no major difference regarding 'FSM' and 'nominal', except that the nominal approach is conceptually related to computer science applications, while we focus on the foundations of mathematics by studying the consistency and inconsistency of various results within the framework of atomic sets.

The motivation for studying FSM also comes from the idea of modelling infinite algebraic structures, that are hierarchically constructed from atoms, in a finitary manner, by analyzing the finite supports of these structures. Thus, in FSM we admit the existence of infinite atomic structures, but for such an infinite structure we remark that only a finite family of its elements (i.e. its 'finite support') is "really important" in order to characterize the related structure, while the other elements are somehow "similar". As an intuitive/straightforward motivation, in a lambda-calculus interpretation, the finite support of a lambda-term is represented by the set of all "free variables" of the term; these variables are those who are really important in order to characterize the term, while the other variables can be renamed (by choosing new names from an infinite family of names) without affecting the essential properties of the lambda-term. This means we can obtain an infinite family of terms starting from an original one (by renaming its bound variables), but in order to characterize this infinite family of terms it is sufficient to analyze the finite set of free variables of the original term. FSM provides important tools for studying infinite algebraic structures in a discrete manner.

Formally, FSM contains both the family of 'non-atomic' ZF sets (which are proved to be trivial FSM sets, i.e. their elements are left unchanged under the effect of the canonical permutation action) and the family of 'atomic' sets (i.e. sets that contain at least an atom somewhere in their structure) with finite supports (hierarchically constructed from the empty set and the fixed ZF set A). One main task now is to analyze whether a classical ZF result (obtained in the framework of non-atomic sets) can be adequately reformulated by replacing 'non-atomic ZF element/set/structure' with 'atomic, finitely supported element/set/structure' in order to be valid also for atomic sets with finite supports. Translating ZF results into FSM is not an easy task. The family of FSM sets is not closed under ZF subsets constructions, meaning that there exist subsets of FSM sets that fail to be finitely supported (for example the simultaneously ZF infinite and coinfinite subsets of the set A). Thus, for proving results in FSM we cannot use related results from the ZF framework without priory reformulating them with respect to the finite support requirement. Furthermore, not even the translation of the results from a non-atomic framework into an atomic framework (such as ZFA obtained by weakening ZF axiom of extensionality) is an easy task. Results from ZF may lose their validity when reformulating them in ZFA. For example, it is known that multiple choice principle and Kurepa's maximal antichain principle are both equivalent to the axiom of choice in ZF. However, Jech proved that multiple choice principle is valid in the Second Fraenkel Model, while the axiom of choice fails in this model.

Furthermore, Kurepa's maximal antichain principle is valid in the Basic Fraenkel Model, while the axiom of choice fails in this model. This means that the following two statements (that are valid in ZF) 'Kurepa's principle implies axiom of choice' and 'Multiple choice principle implies axiom of choice' fail in ZFA. Similarly, there are results that are consistent with ZF, but fail to be consistent with FSM (we particularly mention choice principles and Stone duality).

A proof of an FSM result should be internally consistent in FSM and not retrieved from ZF, that means it should involve only finitely supported constructions (even in the intermediate steps). The meta-theoretical techniques for the translation of a result from non-atomic structures to atomic structures are based on the so called 'S-finite supports principle' claiming that *for any finite set S of atoms, anything that is definable in higher-order logic from S-supported structures by using S-supported constructions is also S-supported.* The formal involvement of the S-finite support principle actually implies a hierarchical construction of the support of a structure by employing the supports of the substructures of a related structure.

The main goal of the book is to provide a set theoretical approach for studying the foundations of finitely supported sets and of related topics. In this sense we analyze the consistency of various forms of choice (and equivalent results), as well as the consistency of results regarding cardinality (cardinals order and cardinals arithmetic), maximality and infinity, in the framework of finitely supported sets. We also introduce finitely supported ordered sets as finitely supported sets that are equipped with finitely supported order relations, and finitely supported algebraic structures as finitely supported sets together with finitely supported internal laws. We provide detailed examples of finitely supported partially ordered sets and finitely supported lattices, and we provided new properties of them. We are particularly focused on fixed point properties of mappings defined on atomic structures and on results regarding various forms of infinity defined within finitely supported sets. Some of these properties are translated from the usual Zermelo Fraenkel framework (of trivially invariant algebraic structures), by replacing '(non-atomic) structure' with '(atomic) finitely supported structure'. However, many properties are specific to the theory of finitely supported sets and lead from the finite support requirement. Some formal definitions are presented below.

A finite set (without other specification) is referred to a set for which there is a bijection with a finite ordinal, i.e. to a set that can be represented as $\{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$. An infinite set (without other specification) means "a set which is not finite". Adjoin to ZF a special infinite set A (called 'the set of atoms' by analogy with ZFA set theory; however, despite classical set theory with atoms, we do not need to modify the axiom of extensionality in order to define A). Actually, atoms are entities whose internal structure is irrelevant (i.e. their internal structure is ignored) and which are considered as basic for a higher-order construction. An invariant set $(X, \cdot)$ is a classical ZF set X equipped with an action $\cdot$ on X of the group of permutations of A, having the additional property that any element $x \in X$ is finitely supported. In a pair $(X, \cdot)$ formed by a ZF set X and a group action $\cdot$ on X of the group of all permutations of A, an arbitrary element $x \in X$ is finitely supported if there exists a finite family $S \subseteq A$ such that any permutation of A that fixes S pointwise also leaves x invariant under the group action $\cdot$. An empty supported element $x \in X$ is called equivariant. If there exists an action $\cdot$ of the group of permutations of A on a set X, then there is an action $\star$ of the group of permutations of A on $\wp(X) = \{Y \,|\, Y \subseteq X\}$, defined by $(\pi, Y) \mapsto \pi \star Y := \{\pi \cdot y \,|\, y \in Y\}$ for all permutations π of A and all $Y \subseteq X$. A subset of X is called finitely supported if it is finitely supported as an element in $\wp(X)$ with respect to the action $\star$. An invariant set is actually an equivariant element at the following order stage in an hierarchical construction. The set of all finitely supported subsets of a finitely supported/invariant set X form a finitely supported/invariant set denoted by $\wp_{fs}(X)$. A subset of X is called uniformly supported is all of its elements are supported by the same finite set of atoms. The Cartesian product of two invariant sets $(X, \cdot)$ and $(Y, \diamond)$ is an invariant set with the action $(\pi, (x, y)) \mapsto (\pi \cdot x, \pi \diamond y)$. Generally, an FSM set is either an invariant set or a finitely supported subset of an invariant set. A relation (or, particularly, a function) between two FSM sets is finitely supported/equivariant if it is finitely supported/equivariant as a subset of the Cartesian product of those two FSM sets. Particularly, a function between two FSM sets X and Y is supported by a finite set S if and only if $f(\pi \cdot x) = \pi \cdot f(x)$, $\pi \cdot x \in X$ and $\pi \cdot f(x) \in Y$ for all $x \in X$ and all permutations π that fix S pointwise. The set of all finitely supported functions from X to Y is denoted by Y^X_{fs}. Whenever, X is a finitely supported/invariant set, we have that the set of all finite, injective tuples of elements from X, denoted by $T_{fin}(X)$, is also a finitely supported/invariant set. Finitely supported partially ordered sets are finitely supported sets equipped with finitely supported partial order relations.

1 Outline of the Book

The book is structured on 14 chapters as presented below.

Details regarding the construction of finitely supported structures, together with a meta-theoretical presentation and several limitations regarding the transferability of the results from ZF to FSM are presented in **Chapter 1**. In the same chapter we establish connections between several frameworks related to finitely supported sets (such as permutative models of ZFA set theory, FM axiomatic set theory, theory of nominal sets, theory of generalized nominal sets, theory of admissible sets and Gandy machines). We describe the methods of translating the results from the non-atomic framework of Zermelo-Fraenkel sets into the atomic framework of sets with finite supports, focusing on the S-finite support principle and on the constructive method of defining supports. We also emphasize the limits of translating non-atomic results into an atomic set theory by presenting examples of valid Zermelo-Fraenkel results that cannot be reformulated using atomic sets.

In **Chapter 2** we formally describe finitely supported sets as classical Zermelo-Fraenkel sets equipped with canonical permutation actions, satisfying a finite support requirement. We provide higher-order constructions of atomic sets (such as powersets, Cartesian products, disjoint unions or function spaces) starting from some basic atomic sets. We present basic properties of finitely supported sets and of mappings between finitely supported sets. We also prove that mappings defined on some specific atomic sets have surprising (fixed points) properties. Particularly, finitely supported self-mappings defined on the finite powerset of atoms have infinitely many fixed points if they satisfy some particular properties (such as strict monotony, injectivity or surjectivity). Finally, we describe Fraenkel-Mostowski axiomatic set theory that is connected with the theory of finitely supported algebraic structures.

The validity and the non-validity of choice principles in various models of Zermelo-Fraenkel set theory and of Zermelo-Fraenkel set theory with atoms (including the symmetric models and the permutation models) was deeply investigated in the last century. Actually, choice principles are proved to be independent from the axioms of Zermelo-Fraenkel set theory and of Zermelo-Fraenkel set theory with atoms, respectively. Since the theory of finitely supported algebraic structures is connected to the related permutation models, it became an important task to study the consistency of choice principles within this new framework. In **Chapter 3** we prove that the choice principles **AC** (axiom of choice), **HP** (Hausdorff maximal principle) **ZL** (Zorn lemma), **DC** (principle of dependent choice), **CC** (principle of countable choice), **PCC** (principle of partial countable choice), **AC(fin)** (axiom of choice for finite sets), **Fin** (principle of Dedekind finiteness), **PIT** (prime ideal theorem), **UFT** (ultrafilter theorem), **OP** (total ordering principle), **KW** (Kinna Wagner selection principle), **OEP** (order extension principle), **SIP** (principle of existence of right inverses for surjective mappings), **FPE** (finite powerset equippolence principle) and **GCH** (generalized continuum hypothesis) are not valid in FSM (i.e. they are in contradiction with the finite support requirement). Proving the inconsistency of choice principles in FSM (i.e. the non-validity of their atomic FSM formulations) is not an easy task because the Zermelo-Fraenkel results between choice principles are not necessarily preserved into this new framework, unless we reprove them with respect to the finite support requirement.

The logicality of the FSM approach is proved in **Chapter 4** by establishing that invariant sets are logical notions in Tarski's sense (i.e. they are left unchanged under the effect of each one-to-one transformation of the universe of all basic elements onto itself). Furthermore, FSM sets also satisfy a weaker form of logicality (i.e. they are invariant under those permutations that fix their support pointwise). We also provide a connection with the Erlangen Program of Felix Klein for the classification of various geometries according to invariants under suitable groups of transformations.

In **Chapter 5** we introduce and study finitely supported partially ordered sets. We study the notions of 'equipollence' of finitely supported mappings and of 'cardinality' of a finitely supported set, proving several properties related to these concepts. For two finitely supported sets X and Y we say that they have the same cardinality, i.e. $|X| = |Y|$, if and only if there exists a finitely supported bijection $f : X \to Y$. Some properties of cardinalities are naturally extended from the non-atomic Zermelo-Fraenkel framework into the world of atomic structures with finite supports. In this sense, we prove that the Cantor theorem and the Cantor-Schröder-Bernstein theorem for cardinalities are still valid in the world of atomic finitely supported sets. Operations on cardinalities such as sum, product and exponential can be defined in FSM. Several other cardinality arithmetic properties are preserved from the classical Zermelo-Fraenkel set theory. However, the dual of the Cantor-Schröder-Bernstein theorem (where cardinalities are ordered via surjective mappings) is no longer valid in this framework. Other specific order properties of cardinalities that do not have related Zermelo-Fraenkel correspondents are also proved. On the family of cardinalities we can define the relations:

- $\leq$: $|X| \leq |Y|$ iff there is a finitely supported injective mapping $f : X \to Y$;
- $\leq^*$: $|X| \leq^* |Y|$ iff there is a finitely supported surjective mapping $f : Y \to X$.

We prove that the relation $\leq$ is equivariant, reflexive, anti-symmetric and transitive, but not total, while the relation $\leq^*$ is equivariant, reflexive and transitive, but not anti-symmetric, nor total.

Several classical fixed points theorems for partially ordered sets, namely Tarski-Kantorovitch Theorem, Bourbaki-Witt Theorem and Kleene-Scott Theorem, are preserved in FSM. These theorems can be consistently reformulated according to the finite support requirement and provide new properties of finitely supported partially ordered sets. Bourbaki-Witt theorem could be used to define recursive data types in FSM and to study computable functions in FSM. Tarski-Kantorovitch theorem and Kleene-Scott theorem could be involved in formal nominal semantics of programming languages, in FSM domain theory, in FSM abstract interpretation, in a FSM theory of iterated function systems. Other fixed point properties for functions defined on invariant sets containing no uniformly supported subsets, including calculability properties for fixed points are proved. Particularly, finitely supported progressive mappings defined on invariant sets containing no infinite uniformly supported subsets have infinitely many fixed points that can be computably described. Specific FSM properties of self-mappings defined on finite powersets or on uniform powersets are presented. We prove that finitely supported order preserving self mappings on the finite powerset and, respectively, on the uniform powerset of a set containing no uniformly supported subsets have least fixed points, and, in some particular cases, such mappings have infinitely many fixed points that can be clearly defined. This is an important extension of Tarski's fixed point theorem for complete lattices that is specific to atomic FSM; generally, in ZF, order-preserving functions on finite powersets do not have fixed points since the finite powersets are not complete lattices. We also present specific fixed point properties for order preserving self mappings defined on the set of atoms, on the finite powerset of atoms, on the powerset of atoms, on the set of all finitely supported functions from A to A, or on higher-order constructions.

In **Chapter 6** we introduce and study lattices in the framework of finitely supported structures. Various properties of lattices are obtained by extending the classical Zermelo-Fraenkel results from the world of non-atomic structures to the world of atomic finitely supported structures. We particularly prove that Tarski fixed point theorem for Zermelo-Fraenkel complete lattices remains valid in the world of finitely supported structures, and we present some calculability properties for specific fixed points of finitely supported monotone self-mappings defined on finitely supported complete lattices. Such results can be applied for the particular finitely supported complete lattices constructed in the next chapter. FSM Tarski fixed point theorem can also be applied in an FSM theory of abstract interpretation to prove the existence of least fixed points for specific finitely supported mappings (defined on invariant complete lattices of properties) modelling the transitions between properties of programming languages. A generalization of FSM Tarski's theorem where the condition of having a least upper bound is imposed only for those uniformly supported subsets of an FSM lattice (and not for all the finitely supported subsets of the related FSM lattice) is also proved.

Some formal results regarding Chapter 5 and Chapter 6 are listed below.

- Let $(X, \sqsubseteq, \cdot)$ be a non-empty finitely supported partially ordered set with the property that every finitely supported totally ordered subset of X has a least upper bound. If $f : X \to X$ is a finitely supported function with the property that $x \sqsubseteq f(x)$ for all $x \in X$, then there exists $x \in X$ such that $f(x) = x$.

- Let $(X, \sqsubseteq, \cdot)$ be a non-empty finitely supported partially ordered set with the property that every uniformly supported subset of X has a least upper bound. Let $f : X \to X$ be a finitely supported function with the property that $x \sqsubseteq f(x)$ for all $x \in X$. Then there exists $x \in X$ such that $f(x) = x$.

- Let $(X, \sqsubseteq, \cdot)$ be a non-empty finitely supported partially ordered set with the property that every uniformly supported subset of X has a least upper bound. Let $f : X \to X$ be a finitely supported order preserving function with the property that there exists $x_0 \in X$ such that $x_0 \sqsubseteq f(x_0)$. Then there exists $x \in X$ with $x_0 \sqsubseteq x$ such that $f(x) = x$.

- Let $(X, \sqsubseteq, \cdot)$ be a finitely supported partially ordered set with the property that every uniformly supported subset has a least upper bound. If $f : X \to X$ is a finitely supported function having the properties that $f(\sqcup Y) = \sqcup f(Y)$ for every uniformly supported subset Y of X and there exist $x_0 \in X$ and $k \in \mathbb{N}^*$ such that $x_0 \sqsubseteq f^k(x_0)$, then f has a fixed point.

- Let $f : \wp_{fin}(A) \to \wp_{fin}(A)$ be finitely supported and strictly order preserving (i.e. f has the property that $X \subsetneq Y$ implies $f(X) \subsetneq f(Y)$). Then we have $X \setminus supp(f) = f(X \setminus supp(f))$ for all $X \in \wp_{fin}(A)$, where A is the set of all atoms and $\wp_{fin}(A)$ is the finite powerset of the set of atoms.

In **Chapter 7** we present various fundamental examples of invariant complete lattices (i.e. lattices that are invariant sets, equipped with invariant partial order relations having the properties that any finitely supported subset has a least upper bound) and analyze their properties. We particularly study the finitely supported subsets of an invariant set, the finitely supported functions from an invariant set to an invariant complete lattice (i.e. the finitely supported L-fuzzy sets with L being an invariant complete lattice), the finitely supported subgroups of an invariant group, and the finitely supported fuzzy subgroups of an invariant group. For this specific invariant complete lattices the theorems presented in the previous chapter can provide new properties.

In **Chapter 8** we define Galois connections between finitely supported ordered structures. Particularly, we present properties of finitely supported Galois connections between invariant complete lattices. As an application, we investigate upper and lower approximations of finitely supported sets using the approximation techniques from the theory of rough sets translated into the framework of atomic sets with finite supports.

In **Chapter 9** we study various FSM forms of infinity (of Tarski type, of Dedekind type, of Mostowski type, and so on), and provide several relationship results between them. An early attempt of presenting various approaches regarding 'infinity' belongs to Tarski who formulates in 1924 some definitions for infinity. The independence of these definitions was later proved in set theory with atoms by Levy. Such independence results can be transferred into classical ZF set theory by employing Jech-Sochor's embedding theorem stating that permutation models of set theory with atoms can be embedded into symmetric models of ZF, and so a statement which holds in a given permutation model of set theory with atoms and whose validity depend only on a certain fragment of that model, also holds in some well-founded model of ZF. In this chapter we emphasized the connections and differences between various definitions of infinity internally in FSM. By presenting examples of atomic sets that satisfy a certain forms of infinity, while they do not satisfy other forms of infinity, we were able to conclude that the FSM definitions of infinity we introduce are pairwise non-equivalent.

We were also interested in FSM uniformly infinite sets that are finitely supported sets containing infinite, uniformly supported subsets. Uniformly supported sets are of interest because they involve boundedness properties of supports, meaning that the support of each element in a uniformly supported set is contained in the same finite set of atoms; in this way, all the individuals in an infinite uniformly supported family can be characterized by involving only the same finitely many characteristics. We proved that the set of all finitely supported functions from A to an FSM set that is not FSM uniformly infinite (i.e. it does not contain an infinite uniformly supported subset) is also not FSM uniformly infinite. In this way several fixed point properties can be obtained. Connections between FSM uniformly finiteness and injectivity/surjectivity of self-mappings on FSM sets are presented. In particular, we prove that for finitely supported self-mappings defined on A, and on the finite powerset of A, respectively, the injectivity is equivalent with the surjectivity. We also discuss the concept of countability in FSM, and present a connection between countable union theorems and countable choice principles. Some formal results of this chapter are summarized below.

Let X be a finitely supported subset of an invariant set Y.

1. X is called *FSM classical infinite* if X does not correspond one-to-one and onto to a finite ordinal, i.e. if X cannot be represented as $\{x_1, \ldots, x_n\}$ for some $n \in \mathbb{N}$. We simply call an FSM classical infinite set as *infinite*, and a set that is not FSM classical infinite as *finite*.

2. X is *FSM covering infinite* if there is a finitely supported directed family $\mathcal{F}$ of finitely supported subsets of Y with the property that X is contained in the union of the members of $\mathcal{F}$, but there does not exist $Z \in \mathcal{F}$ such that $X \subseteq Z$;

3. X is called *FSM Tarski I infinite (TI i)* if there exists a finitely supported one-to-one mapping of X onto $X \times X$, where $X \times X$ is the Cartesian product of X with itself.

4. X is called *FSM Tarski II infinite (TII i)* if there exists a finitely supported family of finitely supported subsets of X, totally ordered by inclusion, having no maximal element.

5. X is called *FSM Tarski III infinite (TIII i)* if if there exists a finitely supported one-to-one mapping of X onto $X + X$, where $X + X$ is the disjoint union of X with itself.

6. X is called *FSM Mostowski infinite (M i)* if there exists an infinite finitely supported totally ordered subset of X.

7. X is called *FSM Dedekind infinite (D i)* if there exists a finitely supported one-to-one mapping of X onto a finitely supported proper subset of X.

8. X is called *FSM ascending infinite (A i)* if there is a finitely supported increasing countable chain of finitely supported sets $X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n \subseteq \ldots$ with $X \subseteq \cup X_n$, but there does not exist $n \in \mathbb{N}$ such that $X \subseteq X_n$;

9. X is called *FSM non-amorphous (N-am)* if X contains two disjoint, infinite, finitely supported subsets.

Some properties of FSM Dedekind infinite sets are listed below:

1. Let X be a finitely supported subset of an invariant set. Then X is FSM Dedekind infinite if and only if there exists a finitely supported one-to-one mapping $f : \mathbb{N} \to X$.

2. Let X be a classical infinite, finitely supported subset of an invariant set. Then $\wp_{fs}(\wp_{fin}(X))$ is FSM Dedekind infinite, where $\wp_{fin}(X)$ is the finite powerset of X.

3. Let X be a finitely supported subset of an invariant set such that X does not contain an infinite uniformly supported subset. Then $\wp_{fin}(X)$ is not FSM Dedekind infinite.

4. The set $\wp_{fs}(A^n)$, where A^n is the n-times Cartesian product of A, does not contain an infinite uniformly supported subset, and so it is not FSM Dedekind infinite, whenever $n \in \mathbb{N}$.

5. Let X be a finitely supported subset of an invariant set such that X does not contain an infinite uniformly supported subset. The set $X^{A^n}_{fs}$ does not contain an infinite uniformly supported subset, and so it is not FSM Dedekind infinite, whenever $n \in \mathbb{N}$.

6. Let X be a finitely supported subset of an invariant set. If $\wp_{fs}(X)$ is not FSM Dedekind infinite, then each finitely supported surjective mapping $f : X \to X$ should be injective. The reverse implication is not valid because any finitely supported surjective mapping $f : \wp_{fin}(A) \to \wp_{fin}(A)$ is also injective, while $\wp_{fs}(\wp_{fin}(A))$ is FSM Dedekind infinite.

7. Let X be a finitely supported subset of an invariant set. If $\wp_{fin}(X)$ is FSM Dedekind infinite, then X should contain two disjoint, infinite, uniformly supported subsets.

8. Let X be a finitely supported subset of an invariant set. If $\wp_{fs}(X)$ is FSM Dedekind infinite, then X contain two disjoint, infinite, finitely supported supported subsets. The reverse implication is not valid.

9. Let X be an FSM Dedekind infinite set. Then there exists a finitely supported surjection $j : X \to \mathbb{N}$. The reverse implication is not valid.

10. Let X be a finitely supported subset of an invariant set. If there exists a finitely supported bijection between X and $X+X$, then X contains an infinite uniformly supported subset. The reverse implication is not valid.

Examples of particular FSM sets satisfying various forms of infinity are presented in the table below.

Set	TI i	TIII i	D i	M i	Asc i	TII i	N-am.
A	No	No	No	No	No	No	No
$A+A$	No	No	No	No	No	No	Yes
$A \times A$	No	No	No	No	No	No	Yes
$\wp_{fin}(A)$	No	No	No	No	Yes	Yes	Yes
$T_{fin}(A)$	No	No	No	No	Yes	Yes	Yes
$\wp_{fs}(A)$	No	No	No	No	Yes	Yes	Yes
$\wp_{fin}(\wp_{fs}(A))$	No	No	No	No	Yes	Yes	Yes
A^A_{fs}	No	No	No	No	Yes	Yes	Yes
$T_{fin}(A)^A_{fs}$	No	No	No	No	Yes	Yes	Yes
$\wp_{fs}(A)^A_{fs}$	No	No	No	No	Yes	Yes	Yes
$A \cup \mathbb{N}$	No	No	Yes	Yes	Yes	Yes	Yes
$A \times \mathbb{N}$	No	Yes	Yes	Yes	Yes	Yes	Yes
$\wp_{fs}(A \cup \mathbb{N})$	No	Yes	Yes	Yes	Yes	Yes	Yes
$\wp_{fs}(\wp_{fs}(A))$	?	Yes	Yes	Yes	Yes	Yes	Yes
$A^{\mathbb{N}}_{fs}$	Yes	Yes	Yes	Yes	Yes	Yes	Yes
$\mathbb{N}^A_{fs}$	Yes	Yes	Yes	Yes	Yes	Yes	Yes

In Figure 1 we present some of the relationships between the FSM definitions of infinite. The 'ultra thick arrows' symbolize *strict* implications (of from p implies q, but q does not imply p), while 'thin dashed arrows' symbolize implications for which we have not proved yet if they are strict or not (the validity of the reverse implications follows when assuming choice principles over non-atomic ZF sets). 'Thick arrows' match equivalences.

In **Chapter 10** we present a large collection of properties of the set of atoms, of its (finite or cofinite) powerset and of its (finite) higher-order powerset in the world of finitely supported algebraic structures. Firstly, we prove that atomic sets have many specific FSM properties (that are not translated from ZF). We can structure these specific properties into five main groups, presenting the relationship between atomic and non-atomic sets, specific finiteness properties of atomic sets, specific (order) properties of cardinalities in FSM, surprising fixed point properties of self-mappings on the (finite) powerset of atoms, and the inconsistency of various choice principles for specific atomic sets.

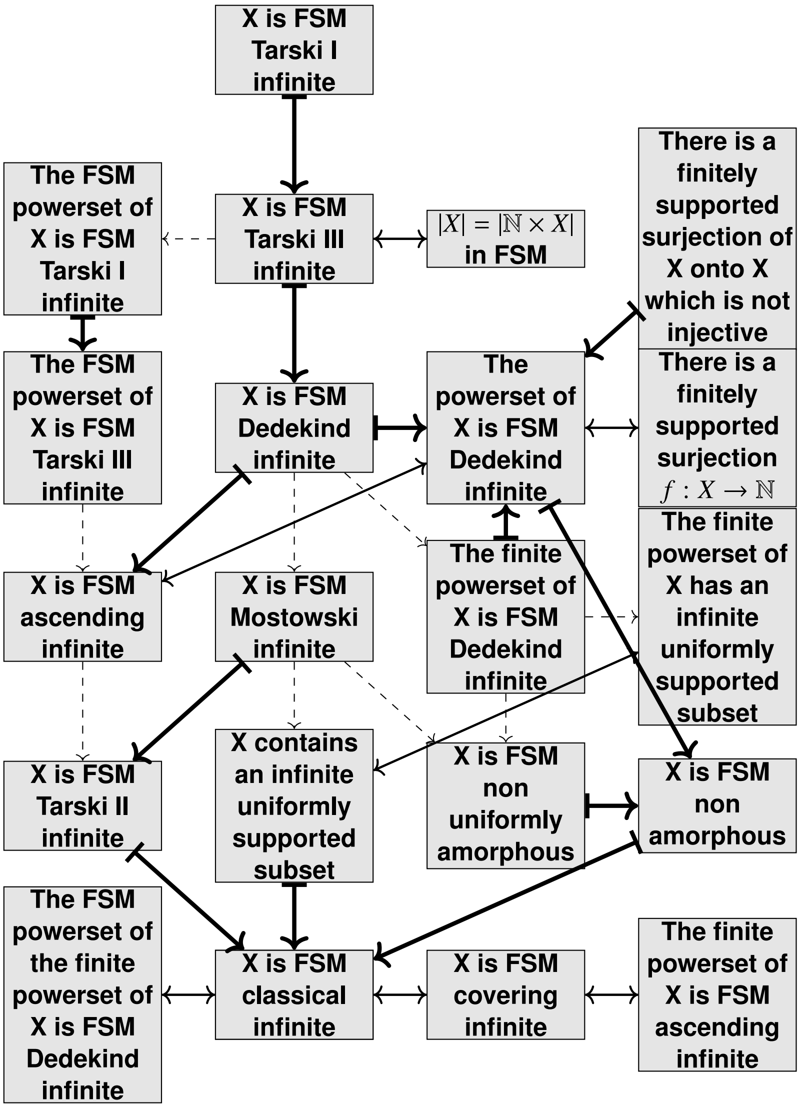


Figure 1: Relationships between various forms of infinity in FSM

Other properties of atoms are obtained by translating classical (non-atomic) ZF results into FSM, by replacing 'non-atomic object' with 'atomic finitely supported object'. Furthermore, we prove that the powerset of atoms satisfies some

References

choice principles such as the Prime Ideal Theorem and the Ultrafilter Theorem, although these principles are generally not valid in FSM. Ramsey Theorem for the set of atoms and Kurepa Antichain Principle for the powerset of atoms also hold, and admit constructive proofs.

In **Chapter 11** we present properties of elements placed outside the support of a given element (named *fresh* elements) and we study a specific quantifier encoding "for all but finitely many" which is placed between $\forall$ and $\exists$.

In **Chapter 12** we present the notion of abstraction appearing in the theory of nominal sets that is used in order to model basic concepts in computer science such as renaming, binding and fresh name. We provide a uniform presentation of the existing results involving abstraction, and emphasized connections with the theory of finitely supported partially ordered sets.

In **Chapter 13** we introduce P_A-sets that are defined as classical sets equipped with actions of the group of all bijections of an amorphous set A. They are constructed in the same way as S_A-sets (sets with permutation actions), except that for defining P_A-sets we consider all the bijections of A, not only the finitary (finitely supported) ones. Furthermore, in contrast to the invariant sets, P_A-sets do not necessarily satisfy the finite support requirement. Relaxed Fraenkel-Mostowski axiomatic set theory represents a refinement of Fraenkel-Mostowski set theory obtained by replacing the finite support axiom with a consequence of it which states that any subset of the set A of atoms is either finite or cofinite. More exactly, the aim of Relaxed Fraenkel-Mostowski set theory is to replace the requirement "finite support for all sets (built on a cumulative hierarchy from a family of basic elements)" with "finite support only for set of basic elements" in order to obtain similar results as in Fraenkel-Mostowski set theory. In this sense, although we do not require the existence of a finite support for any hierarchically defined structure, we prove that several properties of the set of atoms and of the group of all bijections of basic elements (particularly, local finiteness) are preserved. Similarly, we prove that P_A-sets have some properties that are similar to those of invariant sets. We also introduce a mathematics where each set is either finite or cofinite, and we relate it to Relaxed Fraenkel-Mostowski and to Fraenkel-Mostowski set theories.

Chapter 14 presents the conclusions.

This book represents a first set theoretical attempt to make the theory of finitely supported structures accessible to a broad audience. It presents original results regarding choice principles and their equivalences, results regarding cardinalities (trichotomy, Cantor-Bernstein theorem and its dual, cardinalities operations and arithmetic, cardinalities ordering), results regarding several forms of infinity, specific fixed point properties for finitely supported ordered structures, specific properties of atoms (and of functions on atoms and of higher-order constructions on atoms), and the description of various particular invariant complete lattices.