Book Introduction by the Authors

Invited by

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Profinite Semigroups and Symbolic Dynamic

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Most probably, if the reader of this bulletin has ever heard the term "profinite" this is via Eilenberg's theorem on pseudo-varieties of finite monoids. Actually, Birkhoff's theory of varieties had to be reworked in order to suit finite structures. This is precisely what Reitermann did in 1982 by introducing a new type of "equations" or "identities" using operations that no longer belong to the structure, typically $x^{\omega} = x^{\omega+1}$ which can be understood as saying that the finite monoids of the variety have only trivial subgroups. Technically, the idea is based on the notion of free profinite monoid $\widehat{A^*}$ generated by A (and its elements the pseudowords), defined in two equivalent ways: "metrically" as completion of the free monoid for a specific metric (two words are close if they can be distinguished by small finite monoids) and "algebraically" as projective limits of finite monoids in the same spirit that p-adic numbers are the projective limit of the rings $/p^n$. The free monoid A^* is identified with the set of pseudowords having finite (profinite!) length. This publication collects the very last results in the area since the "Finite Semigroups and Universal Algebras" of Jorge Almeida in 1995.

Theoretical computer scientists are familiar with the notion of right infinite, left infinite, two way infinite words and more generally linear structures labelled by finite alphabets. The authors claim that pseudowords are a generalization and that nonfinite pseudowords "start with a right infinite word, end with a left infinite word and have something in the middle" which make them completely distinct from the usual right-, left- or two-way infinite words and invite us to be cautious with the possible misleading interpretation of the symbol ω. This formalism may seem abstract but let me mention a known and old result showing the relevance to the traditional theory of finite automata: there exists an isomorphism between the algebra of clopen subsets of the free profinite monoid and the Boolean algebra of recognizable languages (those recognized by finite automata).

The book starts with an alluring prelude arousing the curiosity of the reader.

In particular and as an illustration of the concepts developed in further chapters, it shows how to use p-adic arithmetic to elegantly prove Skolem-Mahler theorem concerning the zeros of the series associated with rational fractions having rational coefficients. Half of the rest recalls basic definitions and concepts: Hausdorff topological spaces, inverse limits, free groups, finite automata, semigroups, Green relations for semigroups, shift spaces, bifix codes ... The more advanced part investigates the closure in $\widehat{A^*}$ of the uniformly recurrent subsets appearing in minimal shifts and their so-called return words. Most of these shifts are defined by the finite blocks appearing in the fixed points of substitutions of the free monoid, the best known examples of which are the Fibonacci and Thue-Morse words and more generally the DOL-sequences introduced by Lindenmayer. Languages, i.e., subsets of finite words, can be profitably studied by algebraic structures such as congruences and semigroups. This allowed M.P. Schützenberger to characterize the star free regular languages as those languages having only trivial subgroups in the syntactic monoid. In a somewhat similar but sophistigated manner, uniformly recurrent subsets can be associated with so-called Schützenberger groups in $\widehat{A^*}$. The main result, due to the first two authors, shows that a combinatorial property on finite graphs associated with the finite blocks of the shift is a sufficient condition for this group to be a free profinite group. The last chapter has a slightly different focus and works with bifix codes that are factors of some uniformly recurrent subsets. It shows in particular that under some assumptions these codes are finite bases of a free group.

The book is self-contained. Almost all the results are proved in the text otherwise they are referred to some among numerous solved exercices at the end of each chapter. By always choosing the right arguments the proofs are remarkably elegant and appear natural in spite of their technical difficulty.