Interval Temporal Logics: a Journey

Dario Della Monica [*] Valentin Goranko [†]
Angelo Montanari [‡] Guido Sciavicco [§]

Abstract

We discuss a family of modal logics for reasoning about relational structures of intervals over (usually) linear orders, with modal operators associated with the various binary relations between such intervals, known as Allen's interval relations. The formulae of these logics are evaluated at intervals rather than points and the main effect of that semantic feature is substantially higher expressiveness and computational complexity of the interval logics as compared to point-based ones. Without purporting to provide a comprehensive survey of the field, we take the reader to a journey through the main developments in it over the past 10 years and outline some landmark results on expressiveness and (un)decidability of the satisfiability problem for the family of interval logics.

1 Introduction

Temporal reasoning is pervasive in many areas of computer science and artificial intelligence, such as, for instance, formal specification and verification of sequential, concurrent, reactive, real-time systems, temporal knowledge representation, temporal planning and maintenance, theories of actions, events, and fluents, temporal databases, and natural language analysis and processing.

In most cases of temporal reasoning, *time instants (points)* are assumed to be the basic ontological temporal entities. However, often "durationless" time points are not suitable to properly reason about real-world events, which have an intrinsic *duration*. Indeed, many practical aspects of temporality, occurring, for instance, in hardware specifications, real-time processes, and progressive tenses in natural language, are better modeled and dealt with if the underlying temporal ontology is based on *time intervals (periods)*, rather than instants, as the primitive entities.

[*]University of Salerno, Italy, and University of Udine, Italy - ddellamonica@unisa.it
[†]Technical University of Denmark - vfgo@imm.dtu.dk
[‡]University of Udine, Italy - angelo.montanari@uniud.it
[§]University of Murcia, Spain - guido@um.es

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As an example, consider a typical safety requirement of traffic light systems at road intersections as the following one: *'For every time interval I during which the green light is on for the traffic on either road at the intersection, the green light must be continuously off and the red light must be continuously on for the traffic on the other intersecting road, for a time interval beginning strictly before and ending strictly after I.*

The nature of time (in particular, the choice between time instants and time intervals as the primary objects of temporal ontology) has always been a hotly debatable philosophical theme and the philosophical roots of interval-based temporal reasoning can be dated back to Zeno and Aristotle [46]. Already Zeno noted that in an interval-based setting, several of his paradoxes 'disappear' [4], like the flying arrow paradox ("if at each *instant* the flying arrow stands still, how is movement possible?") and the dividing instant dilemma ("if the light is on and it is turned off, what is its state at the *instant* between the two events?").

Of course, the two types of temporal ontologies are closely related and technically reducible to each other: on the one hand, time intervals can be determined by pairs of time instants (*begin–end*); on the other hand, a time instant can be construed as a degenerated 'point interval', whose left and right endpoints coincide. While these reductions can be used to reconcile the different philosophical and ontological standpoints, they do not resolve the main semantic issue arising when developing logical formalisms for capturing temporal reasoning: *should formulae in the given logical language be interpreted as referring to instants or to intervals?*

The possible natural answers to this question lead to (at least) three reasonable alternatives, respectively giving rise to *point-based logics*, *interval-based logics*, and mixed, *two-sorted logics*, where points and intervals are considered as separate sorts on a par and formulae for both sorts are constructed. This exposition is devoted exclusively to the second alternative. The literature on point-based temporal logics is abundant and will not be discussed here. The reader is referred to [4] for a detailed philosophical-logical comparative discussion of both approaches, while a recent study and technical exploration of the two-sorted approach can be found in [3].

One of the first applications of interval-based logical formalisms – to the specification and verification of hardware components – is Propositional Interval Temporal Logic (PITL), introduced by Moszkowski in [45]. An extension of PITL, called *Duration Calculus* (DC), featuring the notion of duration of an event over an interval of time in order to reason about specification and design of time-critical systems, has been actively developed and studied since the early '90s [51]. While DC is one of the most popular and applicable interval-based logical formalisms, its semantics is essentially built on a point-based temporal ontology and thus we will not discuss it here, but we refer the reader to the recent state-of-the-art references on it [35, 50].

An important early work in the formal study of *purely interval-based* temporal ontology and reasoning in AI is [2], where Allen considers the family of binary relations arising between two intervals in a given linear order, subsequently called *Allen's relations*. Besides these, the natural and important operation of *chopping* an interval into two subintervals, giving rise to the ternary interval relation 'chop', was proposed and studied in Moszkowski's work [45].

The systematic *logical* study of purely interval-based temporal reasoning started with the seminal work of Halpern and Shoham [33] (with extended journal version [34]) introducing and analyzing a multi-modal logic, that we will call *Halpern-Shoham logic* (HS for short), featuring one modality for each Allen's relation. Concurrently with [34], Venema introduced and studied the even more expressive interval logic CDT involving binary modal operators associated with the ternary relation Chop (C) and its two residual relations D and T [49]. Decidability and finite axiomatizability issues for CDT fragments have been systematically investigated in [36].

Halpern and Shoham's work initiated a stream of active research on the family $\mathcal{F}$(HS) of fragments of HS, with the main technical issues arising in that research being expressiveness, decidability/undecidability, and complexity of validity and satisfiability. These will be the main themes of the present exposition.

While decidability has been widely assumed to be a standard and expected feature of most (point-based) modal and temporal logics studied and used in computer science, it turned out that *undecidability* is ubiquitous in the realm of interval-based logics. The first such undecidability results were obtained for Propositional Interval Temporal Logic PITL by Moszkowski already in [45]. Furthermore, so sweepingly general undecidability results about HS are given in [34] that for a long time it was considered unsuitable for practical applications and attracted little interest amongst computer scientists. In particular, Halpern and Shoham proved that validity of HS formulae in any class of interval models on linear orders satisfying very weak conditions, including the classes of all linear models, all discrete linear models, and all dense linear models, is undecidable. Moreover, the validities of HS in any of the standard numerical orderings of the natural numbers, integers, and reals (all being Dedekind complete) are not even recursively axiomatizable. Subsequently, the techniques proving such undecidability results were sharpened to apply to a multitude of – sometimes surprisingly simple and inexpressive – fragments of HS, see [8, 28, 37, 38].

The underlying technical reason for these undecidability results can be found in the very nature of purely interval-based temporal reasoning, where all atomic propositions, and therefore all formulae, are interpreted as true or false on every interval, rather than every point, in the model. Thus, the set-theoretic interpretation of an HS formula in an interval model is a set of abstract intervals, that is, a set of pairs of points (a *binary relation*). Thus, HS formulae translate into binary relations over the underlying linear orders, and consequently the validity (resp., satisfiability) problem for HS translates into the respective problem for the *universal (resp., existential) dyadic fragment of second-order logic over linear orders*.

As we already pointed out, for a long time these strong undecidability results have discouraged both search for practical applications and further theoretical research on purely interval-based temporal logics. Meanwhile, several semantic modifications or restrictions, essentially reducing the interval-based semantics to a point-based one, have been proposed to remedy the problem and obtain decidable systems. As an example, already in [45] Moszkowski showed that the decidability of PITL can be recovered by constraining atomic propositions to be point-wise and defining truth of an interval as truth of its initial point (the *locality principle*). The bleak picture started lightening up in the last few years with the discovery of several rather non-trivial cases of decidable fragments of HS; see [16, 18, 23, 43] for some recent accounts and references. Gradually, it became evident that the trade-off between expressiveness and computational affordability in the family $\mathcal{F}$(HS) is rather subtle and sometimes unpredictable, with the border between decidability and undecidability cutting right across the core of that family.

The study and classification of decidable and undecidable fragments of HS has also invoked systematic and comparative analysis of their *expressiveness*. On the one hand, that line of research has led to several correspondence results between fragments of HS and natural fragments of FO; on the other hand, it motivated the classification of the family $\mathcal{F}$(HS) with respect to expressiveness. By systematic use of bisimulations between interval models, we have established a complete set of inter-definability equations between the modal operators of HS, thus obtaining a complete classification of HS fragments with respect to expressiveness [29]. Using that result, we have found that there are exactly 1347 expressively different such fragments out of the $2^{12} = 4096$ subsets of modal operators in HS.

Finally, the strive for obtaining even more expressive, yet decidable interval logics has naturally led to the recently-initiated study of *quantitative* extensions of HS fragments with *metric constraints* on the lengths of intervals, which will be briefly discussed as well.

In this paper we mainly discuss the progress in the field of interval temporal logics over the past 10 years with respect to the topics and developments in which we have been directly involved. It is not a survey but rather travelers' impressions of a long journey, so we make no claim of being all-inclusive or comprehensive.

2 Preliminaries

2.1 Intervals and interval structures

Given a strict partial ordering $\mathbb{D} = \langle D, < \rangle$, an *interval* in $\mathbb{D}$ is an ordered pair $[d_0, d_1]$ such that $d_0, d_1 \in D$ and $d_0 \leq d_1$. A point d *belongs to an interval* $[d_0, d_1]$ if $d_0 \leq d \leq d_1$. If $d_0 < d_1$, then $[d_0, d_1]$ is called a *strict*, or *proper*, interval; otherwise, it is called a *point interval*. The set of all intervals in $\mathbb{D}$, including both strict and point intervals, is usually denoted by $\mathbb{I}(\mathbb{D})^+$, while the set of all strict intervals is denoted by $\mathbb{I}(\mathbb{D})^-$. By $\mathbb{I}(\mathbb{D})$ we will denote either of these. Finally, we call a pair $\langle \mathbb{D}, \mathbb{I}(\mathbb{D}) \rangle$ an *interval structure*.

2.2 Linear orders and interval structures

All interval structures considered here will be assumed to be *linear*, that is, every two points in it are comparable. This restriction can usually be relaxed without essential complications to partial orderings with the *linear interval property*, that is, partial orderings in which every interval is linear. Here is the formal definition in first-order logic:

$$\forall x \forall y (x < y \rightarrow \forall z_1 \forall z_2 (x < z_1 < y \wedge x < z_2 < y \rightarrow z_1 < z_2 \vee z_1 = z_2 \vee z_2 < z_1)),$$

In the figure below an interval structure with the linear interval property is given on the left and an interval structure violating that property is given on the right.

Definition 1. *A linear order, and the associated interval structure, is called:*

- ***finite****, if it has finitely many points;*
- ***unbounded above*** *or* ***to right*** *(resp.,* ***below*** *or* ***to left****), if every point has a successor (resp., predecessor);*
- ***dense****, if between every pair of distinct points there exists another point;*
- ***discrete****, if every point with a successor / predecessor has an immediate successor / predecessor;*
- ***Dedekind complete****, if every non-empty and bounded above set of points has a least upper bound.*

Besides interval logics interpreted in interval structures from the above classes, we will consider interval logics interpreted in single interval structures over the natural orderings of the numerical sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.

2.3 Allen's interval relations

Depicted in Table 1 (first two columns) are all possible binary relations between two strict intervals on a linear order, known as Allen's relations. Besides the identity relation *equal* ($=$), these are (in Allen's original terminology): *before* ($<$), *meets* (m), *overlaps* (o), *finishes* (f), *during* (d), *starts* (s), plus their inverses *later* ($>$), *met-by* (mi), *overlapped-by* (oi), *finished-by* (fi), *contains* (di), *started-by* (si). These 13 relations are *mutually exclusive* and *jointly exhaustive*, meaning that exactly one Allen's relation holds between any given pair of strict intervals.

Each Allen's relation gives rise to a corresponding unary modal operator with Kripke semantics over that relation.

Remark 1. *In [34], Halpern and Shoham have chosen a different notation for Allen's relations from the one used by Allen. For the sake of clarity, in Table 1 we briefly compare the two notations. Note that the semantics of the logic* HS *in Halpern and Shoham's paper is defined including point intervals, but the relations corresponding to the modal operators of* HS *are neither mutually exclusive nor jointly exhaustive there. As an example, in the original semantics of* HS*, both relations* overlaps *and* meets *hold between two intervals* $[a, b]$ *and* $[b, c]$ *with* $a < b < c$*; on the other hand, the intervals* $[a, b]$ *and* $[c, c]$*, with* $b < c$*, are not related by any of Allen's relation.*

While [34] adopts *non-strict* semantics, with point intervals included in the interval structure, in this paper we mainly focus on the *strict* semantics, where these are excluded. This choice conforms to Allen's definition of interval [2] and it has at least two strong motivations. First, a number of representation problems arise when the non-strict semantics is adopted, due to the presence of point intervals, as pointed out in [2]. Second, when point intervals are included, there seems to be no good definition for all interval relations that makes them both pairwise disjoint and jointly exhaustive (see the above remark). On the other hand, while admitting point intervals in the semantics usually strengthens the expressiveness of the modal languages, all known results about decidability and undecidability are invariant with respect to the inclusion or exclusion of point intervals.

An approach avoiding the problems arising in the non-strict semantics was proposed in [3], where both sorts of points and intervals in interval structures are considered on a par, with all natural intra-sort and inter-sort relations arising in the two-sorted universe and the associated with them modal operators.

2.4 Syntax and semantics of Halpern-Shoham's logic HS

The language of HS includes a set of propositional letters $\mathcal{AP}$, the classical propositional connectives $\neg$ and $\vee$ (all others, including the propositional constants $\top$

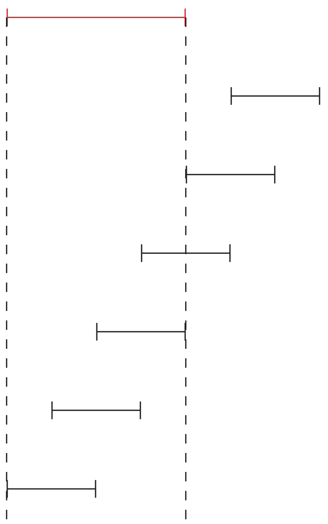
Interval's relations	Allen's notation	HS notation
	*equals* $\{=\}$	
	before $\{<\}$ / *after* $\{>\}$	$\langle L\rangle$ / $\langle\overline{L}\rangle$ (*Later*)
	meets $\{m\}$ / *met-by* $\{mi\}$	$\langle A\rangle$ / $\langle\overline{A}\rangle$ (*After*)
	overlaps $\{o\}$ / *overlapped-by* $\{oi\}$	$\langle O\rangle$ / $\langle\overline{O}\rangle$ (*Overlaps*)
	finished-by $\{fi\}$ / *finishes* $\{f\}$	$\langle E\rangle$ / $\langle\overline{E}\rangle$ (*Ends*)
	contains $\{di\}$ / *during* $\{d\}$	$\langle D\rangle$ / $\langle\overline{D}\rangle$ (*During*)
	started-by $\{si\}$ / *starts* $\{s\}$	$\langle B\rangle$ / $\langle\overline{B}\rangle$ (*Begins*)

Table 1: Relations between pairs of strict intervals.

and $\perp$, are assumed definable as usual), and a family of *interval temporal modal operators (modalities)* of the form $\langle X\rangle$, one for each Allen's relation. Formulae are defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle X\rangle\varphi$$

An *interval model* is a pair $M = \langle \mathbb{D}, V\rangle$, where $V : \mathbb{I}(\mathbb{D}) \to 2^{\mathcal{AP}}$ is a *labeling* assigning to each interval a set of atomic propositions considered true at it.

The *truth of a formula over a given interval* $[a, b]$ *in an interval model* M is defined below by structural induction on formulae. The definition applies both to the strict and the non-strict semantics; however, when point intervals are involved some of Allen's relations and the respective diamond operators trivialize.

- $M, [a, b] \Vdash p$ iff $p \in V([a, b])$, for all $p \in \mathcal{AP}$;
- $M, [a, b] \Vdash \neg\psi$ iff it is not the case that $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \varphi \vee \psi$ iff $M, [a, b] \Vdash \varphi$ or $M, [a, b] \Vdash \psi$;
- $M, [a, b] \Vdash \langle X\rangle\psi$ iff there exists an interval $[c, d]$ such that $[a, b]\ R_X\ [c, d]$, and $M, [c, d] \Vdash \psi$, where R_X is the binary interval relation corresponding to the modal operator $\langle X\rangle$ (Table 1).

More precisely, the semantics of HS is given via the following clauses for the modalities, where referring to an interval $[a, b]$ automatically assumes that $a < b$ in the case of strict semantics and $a \le b$ in the non-strict one.

- $M, [d_0, d_1] \Vdash \langle A \rangle \varphi$ iff $M, [d_1, d_2] \Vdash \varphi$ for some d_2;
- $M, [d_0, d_1] \Vdash \langle L \rangle \varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_1 < d_2$;
- $M, [d_0, d_1] \Vdash \langle B \rangle \varphi$ iff $M, [d_0, d_2] \Vdash \varphi$ for some d_2 such that $d_2 < d_1$;
- $M, [d_0, d_1] \Vdash \langle E \rangle \varphi$ iff $M, [d_2, d_1] \Vdash \varphi$ for some d_2 such that $d_0 < d_2$;
- $M, [d_0, d_1] \Vdash \langle D \rangle \varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_0 < d_2$ and $d_3 < d_1$;
- $M, [d_0, d_1] \Vdash \langle O \rangle \varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_0 < d_2 < d_1 < d_3$;
- $M, [d_0, d_1] \Vdash \langle \overline{A} \rangle \varphi$ iff $M, [d_2, d_0] \Vdash \varphi$ for some d_2;
- $M, [d_0, d_1] \Vdash \langle \overline{L} \rangle \varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_3 < d_0$;
- $M, [d_0, d_1] \Vdash \langle \overline{B} \rangle \varphi$ iff $M, [d_0, d_2] \Vdash \varphi$ for some d_2 such that $d_2 > d_1$;
- $M, [d_0, d_1] \Vdash \langle \overline{E} \rangle \varphi$ iff $M, [d_2, d_1] \Vdash \varphi$ for some d_2 such that $d_2 < d_0$;
- $M, [d_0, d_1] \Vdash \langle \overline{D} \rangle \varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_2 < d_0$ and $d_1 < d_3$;
- $M, [d_0, d_1] \Vdash \langle \overline{O} \rangle \varphi$ iff $M, [d_2, d_3] \Vdash \varphi$ for some d_2, d_3 such that $d_2 < d_0 < d_3 < d_1$.

For each of the above-defined diamond modalities, the corresponding box modality is defined as a dual, e.g., $[A]\varphi \equiv \neg\langle A \rangle \neg\varphi$.

Finally, when the non-strict semantics is assumed, it is natural to consider an additional *modal constant for point intervals*, denoted π, with the following truth definition:

- $M, [d_0, d_1] \Vdash \pi$ iff $d_0 = d_1$.

Validity and *satisfiability* are defined as usual, that is, a formula φ of HS is *satisfiable* if there exists an interval model M and an interval $[a, b]$ such that $M, [a, b] \Vdash \varphi$; φ is *valid*, denoted $\models \varphi$, if it is true on every interval in every interval model. Two formulae φ and ψ are *equivalent*, denoted $\varphi \equiv \psi$, if $\models \varphi \leftrightarrow \psi$.

2.5 Fragments of HS

With every subset $\mathcal{X} = \{\langle X_1\rangle, \ldots, \langle X_k\rangle\}$ of the modal operators of HS we associate the *fragment* $\mathcal{F}_{\mathcal{X}}$ of HS denoted $\mathsf{X_1X_2\ldots X_k}$, with formulae built on the same set of propositional letters $\mathcal{AP}$, but only using modal operators from $\mathcal{X}$. The presence of the superscript $^\pi$ denotes that the modal constant π is added, too. For example, $\mathsf{A\overline{A}}$ denotes the fragment involving the modalities $\langle A\rangle$ and $\langle\overline{A}\rangle$ only, while $\mathsf{A\overline{A}}^\pi$ denotes the fragment involving $\langle A\rangle$, $\langle\overline{A}\rangle$, and π. For any given fragment $\mathcal{F} = \mathsf{X_1X_2\ldots X_k}$ and a modal operator $\langle X\rangle$, we write $\langle X\rangle \in \mathcal{F}$ if $\langle X\rangle \in \{\langle X_1\rangle, \ldots, \langle X_k\rangle\}$. For any given pair of fragments $\mathcal{F}_1$ and $\mathcal{F}_2$, we write $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $\langle X\rangle \in \mathcal{F}_1$ implies $\langle X\rangle \in \mathcal{F}_2$, for every modal operator $\langle X\rangle$.

3 Expressiveness

The study and comparative analysis of the expressiveness of interval logics has been a major research direction in the area. In particular, the natural and important problems arise to identify the mutual definabilities between the modal operators of the logic HS and to classify the fragments of HS with respect to their expressiveness. We will discuss these problems here. In particular, we will present the complete classification of the fragments of HS with respect to their expressiveness in the strict semantics over the class of all linear orders, by identifying a sound and complete set of *inter-definability equations* between the modal operators of HS, summarizing the results presented in [29].

3.1 Expressiveness of HS modalities: some examples

Due to their interval-based interpretation, the modal operators in HS are rather more expressive than what meets the eye. We will only give a couple of testifying examples here:

▹ Using the modality $\langle D\rangle$ corresponding to the sub-interval relation one can express non-trivial combinatorial relationships between width and depth of an interval, of the type:

$$\bigwedge_{i=1}^{d(n)} \langle D\rangle \left(p_i \wedge \bigwedge_{j\neq i} \langle D\rangle \neg p_j \right) \rightarrow \langle D\rangle^n \top$$

for a large enough $d(n)$.

Also, using $\langle D\rangle$ one can express quite special properties of the models, e.g. the formula

$$\langle D\rangle\langle D\rangle\top \wedge [D](\langle D\rangle\top \rightarrow \langle D\rangle\langle D\rangle\top \wedge \langle D\rangle[D]\bot)$$

has neither discrete nor dense models (in the strict semantics), but is satisfiable e.g., in the Cantor space over $\mathbb{R}$.

▹ As proved in [31] the fragment $\mathsf{A}\overline{\mathsf{A}}$ is sufficiently expressive to define all important classes of liner orders mentioned in he previous section, for instance:

- The axioms (SPNL^{der})

  $(\langle A\rangle\langle A\rangle p \to \langle A\rangle\langle A\rangle\langle A\rangle p)$ & $(\langle A\rangle[A]p \to \langle A\rangle\langle A\rangle[A]p)$

  and its inverse (SPNL^{del}) (with $\langle A\rangle$ and $\langle\overline{A}\rangle$ swapped) define the class of *dense* structures, extended with the 2-element linear ordering
  (which cannot be separated in the language of $\mathsf{A}\overline{\mathsf{A}}$).

- The axioms (SPNL^{dir})

  $[A](p\wedge[\overline{A}]\neg p\wedge[A]p) \to [\overline{A}][\overline{A}]\langle A\rangle((\langle A\rangle\neg p\wedge[A][A]p)\vee(\langle A\rangle\top\wedge[A][A]\bot))$,

  and its inverse (SPNL^{dil})

  define the class of *discrete* structures.

- The axiom (SPNL^{c})

  $\langle A\rangle\langle A\rangle[\overline{A}]p \wedge \langle A\rangle[A]\neg[\overline{A}]p \to \langle A\rangle(\langle A\rangle[\overline{A}]\ [\overline{A}]p\wedge\ [A]\ \langle A\rangle\neg\ [\overline{A}]\ p)$

  defines the class of *Dedekind complete* structures.

3.2 Inter-definabilities between HS modalities

Some of the HS modalities are definable in terms of others and for each of the strict and non-strict semantics, we can identify minimal fragments that are expressive enough to define all other operators. For instance:

- In the strict semantics, the six modalities $\langle A\rangle$, $\langle B\rangle$, $\langle E\rangle$, $\langle\overline{A}\rangle$, $\langle\overline{B}\rangle$, $\langle\overline{E}\rangle$ suffice to express all others, as shown by the following equalities [34]:

$$\begin{array}{ll}\langle L\rangle\varphi \equiv \langle A\rangle\langle A\rangle\varphi, & \langle\overline{L}\rangle\varphi \equiv \langle\overline{A}\rangle\langle\overline{A}\rangle\varphi,\\ \langle D\rangle\varphi \equiv \langle B\rangle\langle E\rangle\varphi, & \langle\overline{D}\rangle\varphi \equiv \langle\overline{B}\rangle\langle\overline{E}\rangle\varphi,\\ \langle O\rangle\varphi \equiv \langle E\rangle\langle\overline{B}\rangle\varphi, & \langle\overline{O}\rangle\varphi \equiv \langle B\rangle\langle\overline{E}\rangle\varphi.\end{array}$$

- In the non-strict semantics, the four modalities $\langle B\rangle$, $\langle E\rangle$, $\langle\overline{B}\rangle$, $\langle\overline{E}\rangle$ suffice to express all others, as shown by the following equalities [48]:

$$\begin{aligned}
\langle A\rangle\varphi &\equiv ([E]\bot \wedge (\varphi \vee \langle \overline{B}\rangle\varphi)) \vee \langle E\rangle([E]\bot \wedge (\varphi \vee \langle \overline{B}\rangle\varphi)),\\
\langle \overline{A}\rangle\varphi &\equiv ([B]\bot \wedge (\varphi \vee \langle \overline{E}\rangle\varphi)) \vee \langle B\rangle([B]\bot \wedge (\varphi \vee \langle \overline{E}\rangle\varphi)),\\
\langle L\rangle\varphi &\equiv \langle A\rangle(\langle E\rangle\top \wedge \langle A\rangle\varphi),\\
\langle \overline{L}\rangle\varphi &\equiv \langle \overline{A}\rangle(\langle B\rangle\top \wedge \langle \overline{A}\rangle\varphi),\\
\langle D\rangle\varphi &\equiv \langle B\rangle\langle E\rangle\varphi,\\
\langle \overline{D}\rangle\varphi &\equiv \langle \overline{B}\rangle\langle \overline{E}\rangle\varphi,\\
\langle O\rangle\varphi &\equiv \langle E\rangle(\langle E\rangle\top \wedge \langle \overline{B}\rangle\varphi),\\
\langle \overline{O}\rangle\varphi &\equiv \langle B\rangle(\langle B\rangle\top \wedge \langle \overline{E}\rangle\varphi).
\end{aligned}$$

Also, the modal constant π is definable in terms of $\langle B\rangle$ and $\langle E\rangle$, respectively as $[B]\bot$ and $[E]\bot$.

Furthermore, the presence of π in the language readily embeds the strict semantics into the non-strict one by means of the translation:

- $\tau(p) = p$, for each $p \in \mathcal{AP}$;
- $\tau(\neg\phi) = \neg\tau(\phi)$;
- $\tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi)$;
- $\tau(\langle X\rangle\, \phi) = \langle X\rangle\, (\neg\pi \wedge \tau(\phi))$, for each modality of the language.

3.3 Comparing the expressiveness of fragments of HS

Now, we introduce some formal notions used for comparing the expressiveness of logical languages, adapted to fragments of HS.

Definition 2. *A modal operator $\langle X\rangle$ of* HS *is* definable *in an* HS *fragment $\mathcal{F}$, denoted $\langle X\rangle \lhd \mathcal{F}$, if $\langle X\rangle p \equiv \psi$ for some formula $\psi = \psi(p)$ of $\mathcal{F}$, for any fixed propositional variable p. In such a case, the equivalence $\langle X\rangle p \equiv \psi$ is called an* inter-definability equation for $\langle X\rangle$ in $\mathcal{F}$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be any pair of fragments of HS. We say that:

- $\mathcal{F}_2$ is *at least as expressive as* $\mathcal{F}_1$, denoted $\mathcal{F}_1 \preceq \mathcal{F}_2$, if every operator $\langle X\rangle \in \mathcal{F}_1$ is definable in $\mathcal{F}_2$.
- $\mathcal{F}_1$ is *strictly less expressive* than $\mathcal{F}_2$, denoted $\mathcal{F}_1 \prec \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ but not $\mathcal{F}_2 \preceq \mathcal{F}_1$.

- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *equally expressive* (or, *expressively equivalent*), denoted $\mathcal{F}_1 \equiv \mathcal{F}_2$, if $\mathcal{F}_1 \preceq \mathcal{F}_2$ and $\mathcal{F}_2 \preceq \mathcal{F}_1$.

- $\mathcal{F}_1$ and $\mathcal{F}_2$ are *expressively incomparable*, denoted $\mathcal{F}_1 \not\equiv \mathcal{F}_2$, if neither $\mathcal{F}_1 \preceq \mathcal{F}_2$ nor $\mathcal{F}_2 \preceq \mathcal{F}_1$.

In order to show that $\mathcal{F}_1 \preceq \mathcal{F}_2$, it suffices to prove that every modality of $\mathcal{F}_1$ is definable in $\mathcal{F}_2$, while in order to show that $\mathcal{F}_1 \not\preceq \mathcal{F}_2$, we must show that some modality in $\mathcal{F}_1$ is not definable in $\mathcal{F}_2$.

To show non-definability of a given modal operator in a given fragment, we use a standard technique in modal logic, based on the notion of *bisimulation* and the invariance of modal formulae with respect to bisimulations (see, e.g., [5]). Let $\mathcal{F}$ be an HS fragment. An $\mathcal{F}$-bisimulation between two interval models $M = \langle \mathbb{I}(\mathbb{D}), V \rangle$ and $M' = \langle \mathbb{I}(\mathbb{D}'), V' \rangle$ over $\mathcal{AP}$ is a relation $Z \subseteq \mathbb{I}(\mathbb{D}) \times \mathbb{I}(\mathbb{D}')$ satisfying the following properties:

- *local condition*: Z-related intervals satisfy the same propositional letters over $\mathcal{AP}$;

- *forward condition*: if $([a, b], [a', b']) \in Z$ and $([a, b], [c, d]) \in R_X$ for some $\langle X \rangle \in \mathcal{F}$, then there exists $[c', d']$ such that $([a', b'], [c', d']) \in R_X$ and $([c, d], [c', d']) \in Z$;

- *backward condition*: likewise, but from M' to M.

The important property of bisimulations, used here, is that any $\mathcal{F}$-bisimulation preserves the truth of *all* formulae in $\mathcal{F}$. Thus, in order to prove that an operator $\langle X \rangle$ is not definable in $\mathcal{F}$, it suffices to construct a pair of interval models M and M' and an $\mathcal{F}$-bisimulation between them, relating a pair of intervals $[a, b] \in M$ and $[a', b'] \in M'$, such that $M, [a, b] \Vdash \langle X \rangle p$, while $M', [a', b'] \nVdash \langle X \rangle p$.

3.4 Expressiveness classification of the fragments of HS

As already discussed, in order to classify all fragments of HS with respect to their expressiveness, it suffices to identify all definabilities of modal operators $\langle X \rangle$ in fragments $\mathcal{F}$, where $\langle X \rangle \notin \mathcal{F}$. We say that a definability $\langle X \rangle \lhd \mathcal{F}$ is *optimal* if $\langle X \rangle \ntriangleleft \mathcal{F}'$ for any fragment $\mathcal{F}'$ such that $\mathcal{F}' \prec \mathcal{F}$; a set of definabilities is *optimal* if it consists of optimal definabilities. The rest of the section is devoted to sketching the proof of the following theorem.

Theorem 1 ([29]). *The set of inter-definability equations given in Table 2 is sound, complete, and optimal.*

$\langle L\rangle p \equiv \langle A\rangle\langle A\rangle p$	$\langle L\rangle \lhd \mathsf{A}$
$\langle \overline{L}\rangle p \equiv \langle \overline{A}\rangle\langle \overline{A}\rangle p$	$\langle \overline{L}\rangle \lhd \overline{\mathsf{A}}$
$\langle O\rangle p \equiv \langle E\rangle\langle \overline{B}\rangle p$	$\langle O\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{O}\rangle p \equiv \langle B\rangle\langle \overline{E}\rangle p$	$\langle \overline{O}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$
$\langle D\rangle p \equiv \langle E\rangle\langle B\rangle p$	$\langle D\rangle \lhd \mathsf{BE}$
$\langle \overline{D}\rangle p \equiv \langle \overline{E}\rangle\langle \overline{B}\rangle p$	$\langle \overline{D}\rangle \lhd \overline{\mathsf{BE}}$
$\langle L\rangle p \equiv \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$	$\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$
$\langle \overline{L}\rangle p \equiv \langle \overline{E}\rangle[B]\langle \overline{E}\rangle\langle B\rangle p$	$\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$

Table 2: The complete set of inter-definability equations.

Most of the equations in Table 2 are known from the seminal work of Halpern and Shoham [34], while the definability $\langle L\rangle \lhd \overline{\mathsf{B}}\mathsf{E}$ and its symmetric one, $\langle \overline{L}\rangle \lhd \mathsf{B}\overline{\mathsf{E}}$, are first obtained in [29].

Lemma 1. *The set of inter-definability equations given in Table 2 is sound.*

Proof. As already noted, we only need to prove the soundness for the new inter-definability equation $\langle L\rangle p \equiv \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$ (the proof for the symmetric one defining $\langle \overline{L}\rangle$ is completely analogous, and thus omitted). First, we prove the left-to-right direction. Suppose that $M, [a, b] \Vdash \langle L\rangle p$ for some model M and interval $[a, b]$. This means that there exists an interval $[c, d]$ such that $b < c$ and $M, [c, d] \Vdash p$ (see Figure 1). We exhibit an interval $[a, y]$, with $y > b$ such that, for every x (strictly) in between a and y, the interval $[x, y]$ is such that there exist two points y' and x' such that $y' > y$, $x < x' < y'$, and $[x', y']$ satisfies p. Let y be equal to c. The interval $[a, c]$, which is started by $[a, b]$, is such that for any of its ending intervals, that is, for any interval of the form $[x, c]$, with $a < x$, we have that $x < c < d$ and $M, [c, d] \Vdash p$. As for the other direction, we must show that $\langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$ implies $\langle L\rangle p$. To this end, suppose that $M, [a, b] \Vdash \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$ for a model M and an interval $[a, b]$. Then, there exists an interval $[a, c]$, for some $c > b$ such that $[E]\langle \overline{B}\rangle\langle E\rangle p$ is true on $[a, c]$ (see Figure 1). As a consequence, the interval $[b, c]$ must satisfy $\langle \overline{B}\rangle\langle E\rangle p$, that means that there are two points x and y such that $y > c$, $b < x < y$, and $[x, y]$ satisfies p. Since $x > b$, then $M, [a, b] \Vdash \langle L\rangle p$. $\square$

Proving the completeness is the hard task; optimality is established together with it. In the following, we provide a general overview of the proof idea. A detailed sketch of the proof of Theorem 1 is presented in [29] and the complete proof with all technical details can be found in [28].

For each HS operator $\langle X\rangle$, we show that $\langle X\rangle$ is not definable in any fragment of HS that does not contain $\langle X\rangle$ and does not contain as definable (according to Table 2) all operators of some of the fragments in which $\langle X\rangle$ is definable (accord-

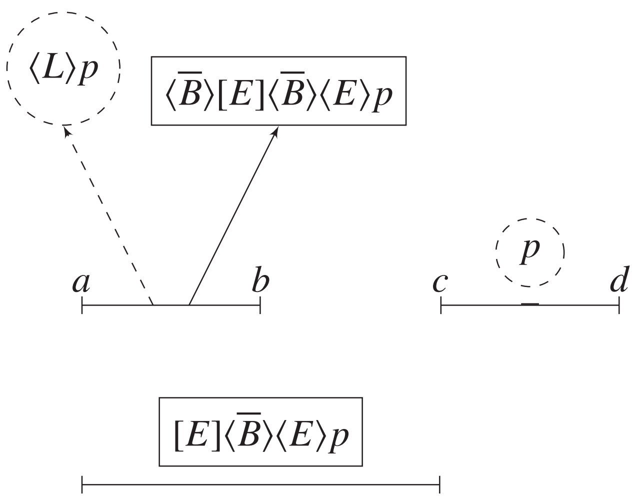


Figure 1: $\langle L\rangle p \equiv \langle \overline{B}\rangle[E]\langle \overline{B}\rangle\langle E\rangle p$.

ing to Table 2). More formally, for each HS operator $\langle X\rangle$, the proof consists of the following steps:

1. Using Table 2, identify all fragments $\mathcal{F}_i$ such that $\langle X\rangle \lhd \mathcal{F}_i$.

2. Produce the list $\mathcal{M}_1, \ldots, \mathcal{M}_m$ of all $\subseteq$-maximal fragments of HS that contain neither the operator $\langle X\rangle$ nor any of the fragments $\mathcal{F}_i$ identified by the previous step;

3. For each fragment $\mathcal{M}_i$, for $i \in \{1, \ldots, m\}$, provide a bisimulation for $\mathcal{M}_i$ that is not a bisimulation for X.

3.5 Expressiveness classification: summary

We have used the equations in Table 2 as the basis of a simple computer program that identifies and counts all expressively different fragments of HS with respect to the strict semantics on the class of all linear orders. Using that program, we have established that there are exactly 1347 expressively different such fragments of HS, out of the $2^{12} = 4096$ subsets of HS modalities.

We emphasize that not all inter-definability equations listed in Table 2, neither the resulting classification, apply in the non-strict semantics. For instance, as shown in [48] that in the non-strict semantics $\langle A\rangle$ (resp., $\langle \overline{A}\rangle$) can be defined in $\overline{\mathsf{B}}\mathsf{E}$ (resp., $\mathsf{B}\overline{\mathsf{E}}$). Moreover, the completeness of the set of equations in Table 2 need not hold any longer if the semantics is restricted to specific classes of linear orders. For instance, in discrete linear orders, $\langle A\rangle$ can be defined in $\overline{\mathsf{B}}\mathsf{E}$ as follows: $\langle A\rangle p \equiv \varphi(p) \vee \langle E\rangle\varphi(p)$, where $\varphi(p)$ is a shorthand for $[E]\bot \wedge \langle \overline{B}\rangle([E][E]\bot \wedge \langle E\rangle(p \vee \langle \overline{B}\rangle p))$; likewise, $\langle \overline{A}\rangle$ is definable in $\mathsf{B}\overline{\mathsf{E}}$. As another example, in dense linear orders, $\langle L\rangle$ can be defined in DO as $\langle L\rangle p \equiv \langle O\rangle(\langle O\rangle\top \wedge [O](\langle O\rangle p \vee \langle D\rangle p \vee \langle D\rangle\langle O\rangle p))$; likewise, $\langle \overline{L}\rangle$ is definable in $\overline{\mathsf{D}}\overline{\mathsf{O}}$.

4 Deciding Satisfiability

Perhaps the currently most challenging, still open problem in the area of interval temporal logics is to obtain a complete classification of the fragments of HS with respect to decidability/undecidability of their satisfiability problem. In particular, we are interested in identifying all maximally expressive, yet decidable such fragments. In this section, we outline the decidability/undecidability landscape in the family of the fragments of HS and discuss the general techniques, used so far for proving decidability and undecidability of satisfiability for these fragments.

A complete picture of the state of the art about the classification of HS fragments with respect to the satisfiability problem can be found in [28, Appendix A]. Besides, a collection of web tools is available on the website `http://itl.dimi.uniud.it/content/logic-hs`, that can be used to identify the status (decidable/undecidable/unknown yet) of the satisfiability problem of any specific fragment, over several classes of linear orders (all, dense, discrete, and finite) in both strict and non-strict semantics, as well as to compare relative expressive power of any pair of HS fragments.

4.1 Overview of decidability methods and results

The early decidability results about interval logics were based on radical restrictions of the interval-based semantics, essentially reducing it to a point-based one. Such restrictions include *locality*, according to which all atomic propositions are evaluated point-wise, meaning that their truth over an interval is defined as truth at its initial point, and *homogeneity*, according to which truth of a formula over an interval implies truth of that formula over every sub-interval. By imposing such constraints, decidability of interval logics can be proved by embedding it into a suitable point-based temporal logic, as in [45, 48]. Decidability can also be achieved by constraining the class of temporal structures over which the logic is interpreted. This is the case with *split-structures*, where any interval can be "chopped" in at most one way. The decidability of various interval logics, including HS, interpreted over split-structures, has been proved by embedding them into decidable first-order theories of time granularities [44].

For some simple fragments of HS, like $\mathsf{B}\overline{\mathsf{B}}$ and $\mathsf{E}\overline{\mathsf{E}}$, decidability can be obtained immediately and without any semantic restriction, by means of direct translation to the point-based semantics and reduction to decidability of respective point-based temporal logics [32]. In any of these logics, one of the endpoints of every interval related to the current one remains fixed, thereby reducing the interval-based semantics to the point-based one by mapping every interval of the generated sub-model to its non-fixed endpoint. Consequently, these fragments can be polynomially translated to the basic temporal logic with Future and Past

$\mathsf{TL}[\mathsf{F},\mathsf{P}]$, thus proving their NP-completeness when interpreted on the class of all linearly ordered sets or on any of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ [30, 32].

We note that most of the fragments of HS are sufficiently expressive to force infinity of an interval structure, and therefore the standard approach to proving decidability in modal logic based on recursive axiomatization plus finite model property is not applicable here. Automata-based methods, based e.g. on Büchi and Rabin theorems (implying decidability of MSO theories of various linear orders and trees), do not apply either, because, as mentioned earlier, satisfiability and validity in interval logics are *dyadic*, not monadic, second-order properties. Thus, new approaches for obtaining decidability results for fragments of HS with unrestricted and genuinely interval-based semantics, non-reducible to point-based one, were needed.

The first such decidability results are obtained in the early 2000s by means of suitable *translations* to other logics, already known to be decidable over linear orders. Such a translation is constructed for the fragment $\mathsf{A}\overline{\mathsf{A}}$, also known as *Propositional Neighborhood Logic* (PNL) [31] into the two-variable fragment of first-order logic with uninterpreted binary relations over linear domains $\mathsf{FO}^2[=,<]$. Thus, decidability, in NEXPTIME, of PNL is obtained in [17, 18] by reduction to the NEXPTIME-complete decidability result for $\mathsf{FO}^2[=,<]$ due to Otto [47]. In fact, the satisfiability problem for PNL turns out to be NEXPTIME-complete, too, by translation from $\mathsf{FO}^2[=,<]$ back to PNL^π in the non-strict semantics, thus implying that the latter logical language is expressively equivalent to the former. Otto's results, and consequently the decidability of PNL, apply not only to the class of all linear orders, but also to some natural sub-classes of it, such as the class of all finite linear orders, the class of all well-founded linear orders, and $\mathbb{N}$.

The so far most fruitful and widely applicable method for obtaining decidability results and decision procedures for fragments of HS not reducible to point-based logics has been the method of *semantic tableau*, often combined with a *(bounded) pseudo-model property*. The method of semantic tableau consists in developing sound, complete, and terminating procedures for tableau-based search of a finite, satisfying the input formula "pseudo-model". Pseudo-models are abstract finite Hintikka-type structures that can be obtained from (possibly infinite) interval structures by filtration-like constructions, specific to the fragment under consideration, that preserve truth of formulae from that fragment, so that a formula is satisfiable if and only if there is a pseudo-model that satisfies it.

This method has been successfully applied for instance to the fragment D, with modality associated with the (strict) sub-interval relation, interpreted over dense linear orderings [14, 15, 16]. In Figure 2 we illustrate a typical pseudo-model (on the left) for the fragment D that corresponds to an interval structure (on the right) over the ordering of the rationals $\mathbb{Q}$. The irreflexive nodes of this pseudo-model represent single intervals while the reflexive ones represent infinite clusters

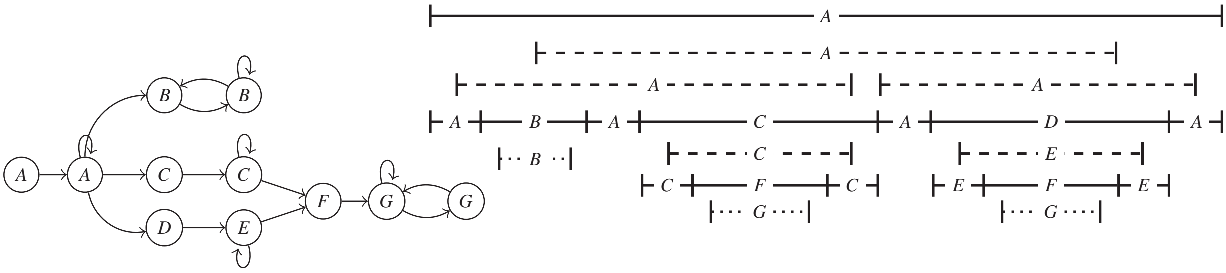


Figure 2: An example of a finite pseudo-model for D (on the left) and its corresponding interval D-structure (on the right) on the ordering of the rationals $\mathbb{Q}$.

(layers, or 'cushions') of (strict) sub-intervals satisfying the same subformulae of the input formula.

The method is subsequently extended to the (maximal decidable) fragment $\mathsf{B\overline{B}D\overline{D}L\overline{L}}$ (and, by symmetry, $\mathsf{E\overline{E}D\overline{D}L\overline{L}}$), interpreted over $\mathbb{Q}$ [40, 41].

In order to establish upper complexity bounds, or sometimes even to ensure termination of the tableau method, a bound of the size of the satisfying pseudo-model has to be established. A method for obtaining pseudo-models of bounded size consists in removing "redundant" points and intervals from an initial finite, or finitely presentable (e.g. periodic) pseudo-model. Using tableau-based method and pseudo-model size-reducing techniques, the earlier mentioned decidability result for PNL is independently re-established and extended in [20, 21, 25], where optimal tableau-based decision procedures for PNL and its future fragment RPNL are developed for several different classes of orderings. More recent work extends these decidability results to $\mathsf{AB\overline{B}\overline{L}}$ [24, 26] (and to $\mathsf{\overline{A}E\overline{E}L}$ by symmetry) and, on finite linear orderings, to $\mathsf{AB\overline{B}\overline{A}}$ (and, by symmetry, to $\mathsf{AE\overline{E}\overline{A}}$) [42].

4.2 Overview of undecidability methods and results

The first undecidability results for HS validity and satisfiability come from the original work of Halpern and Shoham [34] and cover almost all interesting classes of linearly ordered sets:

Theorem 2 ([34])**.** *The validity problem for* HS *is undecidable (r.e. hard) over any class of linear orderings that contains at least one linear ordering with an infinite ascending or descending sequence of points.*

In particular, this result applies to all natural unbounded time-flows such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. The proof is by reduction from the non-halting problem for Turing machines, involving a quite ingenious encoding of Turing machine configurations into unbounded interval structures.

Under a natural additional assumption, Halpern and Shoham show that the undecidability can be much worse:

Theorem 3 ([34]). *The validity in* HS *over any class of Dedekind complete ordered structures containing at least one with an infinitely ascending sequence is* Π^1_1*-hard.*

In particular, the validity in HS over any of the orderings of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ is not recursively axiomatizable. The proof is by reduction from the problem of existence of a computation of a given non-deterministic Turing machine that enters the initial state infinitely often to testing satisfiability in HS.

Later, Lodaya proved that the rather small fragment BE is sufficiently expressive to carry out Halpern and Shoham's idea of encoding Turing machine configurations and consequently, to yield undecidability [37].

More recently, a number of other HS fragments have been proved undecidable [8, 9, 11, 12, 28, 38, 39] by means of suitable reductions from known undecidable problems. The most widely applied such reductions have been constructed from several variants of the *tiling problem*: the $\mathbb{N}\times\mathbb{N}$ tiling problem [8], the octant tiling problem [8, 9, 11], and the finite tiling problem [12].

In the following, we outline the idea underlying the reduction from the octant tiling problem, which is the problem of establishing whether a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ can tile the 2nd octant of the integer plane $O = \{(i, j) : i, j \in \mathbb{N} \wedge 0 \leq i \leq j\}$. This problem can be easily related to interval structures because points in O are naturally interpretable as intervals on $\mathbb{N}$.

Now, the technical details. For every tile type $t_i \in \mathcal{T}$, let $right(t_i)$, $left(t_i)$, $up(t_i)$, and $down(t_i)$ be the colors of the corresponding sides of t_i. To solve the problem, one must find a function $f : O \to \mathcal{T}$ such that

$$right(f(n, m)) = left(f(n + 1, m))$$

and

$$up(f(n, m)) = down(f(n, m + 1)).$$

The undecidability of the tiling problem for O is proved in [6] from that of the tiling problem for $\mathbb{Z} \times \mathbb{Z}$ (known to be co-r.e. complete by a reduction from the halting problem of a Turing machine), through the tiling problem for $\mathbb{N} \times \mathbb{N}$, by application of König's Lemma.

Given an instance of the octant tiling problem OTP($\mathcal{T}$), where $\mathcal{T}$ is the finite set of tiles types, a reduction from OTP($\mathcal{T}$) to the satisfiability problem for a logic $\mathcal{L}$ consists of the construction of a formula $\Phi_{\mathcal{T}}$, parametric in $\mathcal{T}$ and belonging to the language of $\mathcal{L}$, such that $\Phi_{\mathcal{T}}$ is satisfiable if and only if $\mathcal{T}$ tiles O.

Let $\mathcal{T} = \{t_1, \ldots, t_k\}$ be an arbitrary finite set of tile types. We assume the set of atomic propositions $\mathcal{AP}$ to be finite (but arbitrary) and to contain, inter alia, the following propositional variables: u, $*$, Id, tile, $\mathtt{t}_1, \ldots, \mathtt{t}_k$, and up_rel. The general idea of the encoding is the following. First, for any given HS fragment $\mathcal{L}$ and any

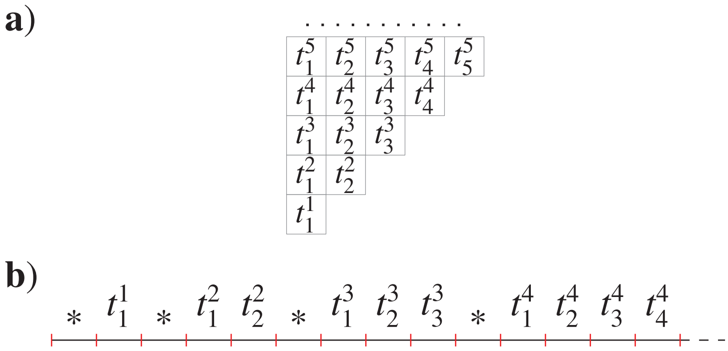


Figure 3: The encoding of the octant tiling problem: a) the cartesian representation, and b) the corresponding interval representation.

starting interval $[a, b]$, we consider a (possibly infinite) set of intervals $\mathcal{G}_{[a,b]}$ that can be 'reached' by means of the modalities of $\mathcal{L}$ starting from $[a, b]$. The set $\mathcal{G}_{[a,b]}$ can be viewed as the universe of intervals on which we work. Then, we exploit the modalities of $\mathcal{L}$ to define a *global modal operator* $[G]$ such that $[G]\varphi$ holds over the interval $[a, b]$ if and only if φ holds over each interval in $\mathcal{G}_{[a,b]}$.

The proof is based on the following main steps:

- definition of the u-*chain*: we set our framework by forcing the existence of a unique infinite chain of u-intervals (u-chain, for short) on the linear ordering. They will be used as cells to arrange the tiling. We also have to provide a way to step from an u-interval to its immediate successor in the chain;

- definition of the Id-*chain*: the octant is encoded by means of a unique infinite sequence of Id-intervals (Id-chain, for short), each of them representing a row of the octant. An Id-interval is composed by a sequence of u-intervals; each u-interval is used either to represent a part of the plane or to separate two rows. In the former case it is labelled with tile, while in the latter case it is labelled with $*$;

- encoding of the *above-neighbor* and *right-neighbor* relations, connecting each tile in the octant with, respectively, the one immediately above it and the one at its right, if any. The encoding of such relations must be done in such a way that the following *commutativity property* holds: *given any two* tile-*intervals* $[c, d]$ *and* $[e, f]$, *if there exists a* tile-*interval* $[d_1, e_1]$, *such that* $[c, d]$ *is right-connected to* $[d_1, e_1]$ *and* $[d_1, e_1]$ *is above-connected to* $[e, f]$, *then there also exists a* tile-*interval* $[d_2, e_2]$ *such that* $[c, d]$ *is above-connected to* $[d_2, e_2]$ *and* $[d_2, e_2]$ *is right-connected to* $[e, f]$.

A generic encoding of the octant tiling problem is depicted in Figure 3.

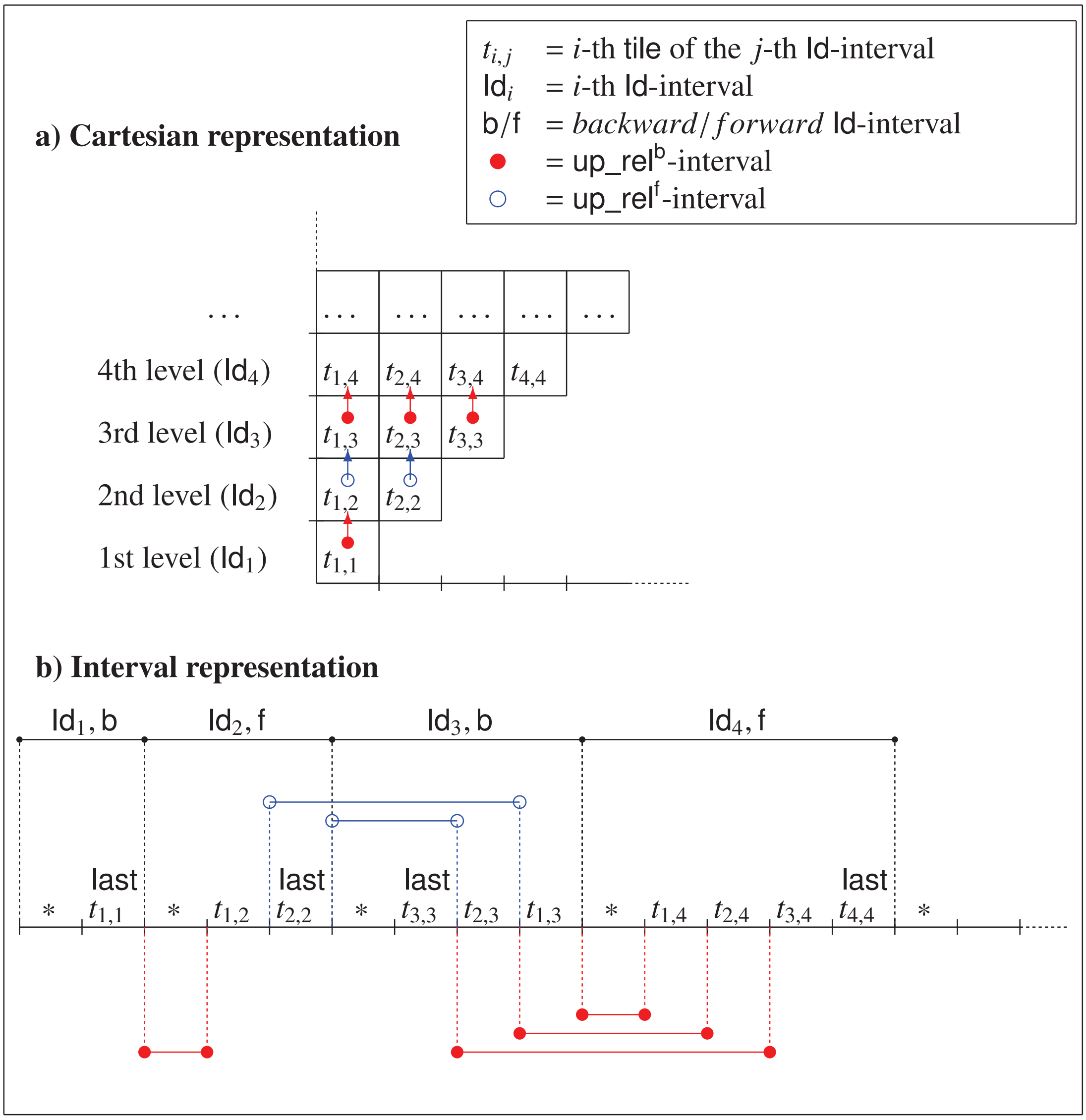


Figure 4: The *above-neighbor* relation encoded in the fragment AO.

The above-described framework is basically the same for all the reductions from variants of the tiling problem. The main difference, and the main difficulty of the reduction, comes from the very limited expressiveness of the fragment under consideration (a number of minimal undecidable HS fragments featuring one or two modalities have been identified). For each different fragment, specific technical tricks are needed, making use of additional propositional letters besides the above-mentioned ones.

As an example, in Figure 4, we show the encoding of the above-neighbor relation for the OTP($\mathcal{T}$) in the HS fragment AO, whose modalities correspond to Allen's relations *meets* and *overlaps* [9].

Lastly, strong and rather unexpected undecidability results have been obtained in [39] and [38] for the HS fragments BD and D, respectively, by means of a reduction from the halting problem for two-counter automata.

5 Metric and spatial extensions

Both interval structures and interval logics are amenable to various natural extensions. In this section, we briefly discuss two of them:

1. *metric interval logics*, based on interval structures over linear orders endowed with distance between points, and thus with a natural notion of *interval length*, and with language extended with arithmetic constraints on interval length;

2. *spatial interval logics*, extending the one-dimensional interval structures to two- and more- dimensional spatial structures.

5.1 Metric interval temporal logics

The idea of adding metric features to point-based temporal logics has been explored in several ways, but metric extensions of purely interval-based logics have only been developed and investigated quite recently, so far mainly on interval structures over the natural numbers.

In [19], Bresolin et al. introduce and study a family of metric extensions of the HS fragment A, also known as *Right* PNL (RPNL for short), with a special attention to decidability and expressive completeness issues. Such a work has been subsequently extended to the family of metric extensions of the full PNL [7, 10]. The most expressive language in that family, called *Metric* PNL (MPNL, for short) features a set of special atomic propositions representing integer constraints (equalities and inequalities) on the length of the intervals over which they are evaluated. In [7, 10], MPNL has been proved to be decidable in 2NEXPTIME, and EXPSPACE-hard and particularly suitable for dealing with metric constraints, thus emerging as a viable alternative to existing logical systems for quantitative temporal reasoning.

In [22], decidability of MPNL has been extended to the class of interval structures over finite linear orders and to $\mathbb{Z}$. Moreover, an optimal decision procedure running in EXPSPACE is provided, thus proving that the satisfiability problem for MPNL over finite linear orders (resp., $\mathbb{N}$, $\mathbb{Z}$) is EXPSPACE-complete.

5.2 Spatial generalization of metric interval logics

The transfer of formalisms, techniques, and results from the temporal context to the spatial one is quite common in computer science. However, it (almost) never comes for free: it usually involves a blow up in complexity, that can possibly yield undecidability.

The main goal of spatial formal systems is to capture common-sense knowledge about space and to provide a calculus of spatial information. Information about spatial objects may concern their shape and size, the distance between them, their topological and directional relations. Depending on the considered class of spatial relations, we can distinguish between *topological* and *directional* spatial reasoning. While topological relations between pairs of spatial objects (viewed as sets of points) are preserved under translation, scaling, and rotation, directional relations depend on the relative spatial position of the objects. A comprehensive and up-to-date survey on topological, directional, and combined constraint systems and relations can be found in [1, 27].

In [13], Bresolin et al. investigate a two-dimensional variant of metric RPNL, called the Directional Area Calculus (DAC). DAC allows one to reason with basic shapes, such as lines, points, and rectangles, directional relations, and (a weak form of) areas. It features two modal operators: *somewhere to the north* and *somewhere to the east*. Moreover, by means of special *atomic propositions*, it makes it possible to constrain the length of the horizontal (resp., vertical) projections of objects. Despite its simplicity, DAC allows one to express meaningful spatial properties. As an example, combining horizontal and vertical length constraints, conditions like "*the area of the current object is less than 4 square meters*" can be expressed in DAC. The satisfiability problem for DAC has been proved to be decidable in 2NEXPTIME [13]. In the same paper, the authors also study a proper fragment of DAC, called Weak DAC (WDAC), which is expressive enough to capture meaningful qualitative and quantitative spatial properties. Decidability of WDAC is proved by a decision procedure whose complexity is exponentially lower than that for DAC. Optimality is an open issue for both DAC and WDAC.

6 Concluding remarks: the roads ahead

Despite the very substantial progress over the past 10 years in the research area of interval temporal logics, the field is still very rich with interesting challenges and unexplored paths. Here we will outline our present view of the main immediate and long-term challenges in the field.

The main items in the current research agenda are:

- extending the expressiveness classification result for the family of fragments of HS from [29] to the non-strict semantics and to the most important classes of linear orders (e.g., finite, discrete, dense, etc.);

- obtaining a complete classification of the family of HS fragments with respect to decidability/undecidability of their satisfiability problem, first on the class of all interval structures over linear orders, and then on the important subclasses of it. Currently, more than 90% of these fragments have already been classified (for a summary of the current state of the classification, see the web page `https://itl.dimi.uniud.it/content/logic-hs`), but the remaining cases are expected to be the most difficult to settle;

- extending the study of metric extensions of interval logics from PNL to other important fragments of HS, and over other important metrizable linear orders, notably $\mathbb{Q}$.

The long-term research perspectives in the field include:

- quest for automata-based techniques for proving decidability of interval logics;
- development of methods and algorithms for model-checking in finitely presentable infinite interval structures, such as ultimately periodic ones.
- last but not least, identifying and developing major applications of interval logics studied here, that would justify and reward the sustained research investment presented here.

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