Addenda to the Survey of Layout Problems

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Abstract

In 2002, J. Díaz, M. Serna and the author published "A Survey of Graph Layout Problems", which then was a complete view of the current state of the art of layout problems from an algorithmic point of view. The current review expands the contents of the original survey with updated results from these latest ten years and contributes an extensive bibliography.

1 Introduction

Graph layout problems are a particular class of combinatorial optimization problems whose goal is to find a linear layout of an input graph in such way that a certain objective cost is optimized. In [Díaz et al., 2002], Maria Serna, Josep Díaz and me presented a survey that considered their motivation, complexity, approximation properties, upper and lower bounds, heuristics and probabilistic analysis on random graphs. The result was a complete view of the current state of the art with respect to layout problems from an algorithmic point of view.

In the occasion of Professor Josep Díaz's 60th birthday, a conference on Trends in Theoretical Computer Science was organized [Àlvarez et al., 2011]. Besides celebrating his birthday, the purpose of the conference was to bring together researchers to discuss various aspects of Theoretical Computer Science. As invited speaker, I opted to take the opportunity to revisit layout problems —the topic of my thesis and one of the favorites themes of Professor Díaz.

Indeed, after ten years, I found worthwhile to perform an exhaustive search on the newer literature on the topic and, consequently, to update the results in the survey. I was also curious to know whether major results in heuristics for the Minimum Linear Arrangement problem had appeared. Furthermore, the current status of the open problems stated in the original survey intrigued me.

The result of this search is synthesised in these addenda to the survey. In particular, the following topics are covered: new **NP**-completeness results, new polynomial time solvable graph classes, new fixed parametrized complexity results,

new approximability results, and new practical results on the Minimum Linear Arrangement problem. It does not cover: new application domains, new results on random graphs, other variants of layout problems.

2   Layout problems

For the sake of completeness, we start with the definition of several graph layout problems and associated concepts. This enables us to treat different layout problems using a unique framework.

Linear layouts. A *linear layout*, or simply a *layout*, of an undirected graph $G = (V, E)$ with $n = |V|$ vertices is a bijective function $\varphi \colon V \to [n] = \{1, \ldots, n\}$. A layout has also been called a *linear ordering* [Adolphson et al., 1973], a *linear arrangement* [Shiloach, 1979], a *numbering* [Chinn et al., 1982] or a *labeling* [Juvan et al., 1992] of the vertices of a graph. We denote by $\Phi(G)$ the set of all layouts of a graph G.

Given a layout φ of a graph $G = (V, E)$, its *reversed layout* is denoted φ^R and is defined by $\varphi^R(u) = |V| - \varphi(u) + 1$ for all $u \in V$.

Layout measures. Given a layout φ of a graph $G = (V, E)$ and an integer i, define the set $L(i, \varphi, G) = \{u \in V \mid \varphi(u) \leq i\}$ and the set $R(i, \varphi, G) = \{u \in V \mid \varphi(u) > i\}$. The *edge cut* at position i of φ is defined as

$$\theta(i, \varphi, G) = |\{uv \in E \mid u \in L(i, \varphi, G) \land v \in R(i, \varphi, G)\}|$$

and the *modified edge cut* at position i of φ as

$$\zeta(i, \varphi, G) = |\{uv \in E \mid u \in L(i, \varphi, G) \land v \in R(i, \varphi, G) \land \varphi(u) \neq i\}|.$$

The *vertex cut* or *separation* at position i of φ is defined as

$$\delta(i, \varphi, G) = |\{u \in L(i, \varphi, G) \mid \exists v \in R(i, \varphi, G) : uv \in E\}|.$$

Given a layout φ of G and an edge $uv \in E$, the *length* of uv on φ is

$$\lambda(uv, \varphi, G) = |\varphi(u) - \varphi(v)|.$$

These measures are summarized in Table 1.

Table 1: Layout measures for a layout φ of a graph $G = (V, E)$.

$$
\begin{aligned}
L(i, \varphi, G) &= \{u \in V \mid \varphi(u) \le i\}. \\
R(i, \varphi, G) &= \{u \in V \mid \varphi(u) > i\}. \\[4pt]
\theta(i, \varphi, G) &= |\{uv \in E \mid u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G)\}|. \\
\zeta(i, \varphi, G) &= |\{uv \in E \mid u \in L(i, \varphi, G) \wedge v \in R(i, \varphi, G) \wedge \varphi(u) \ne i\}|. \\
\delta(i, \varphi, G) &= |\{u \in L(i, \varphi, G) \mid \exists v \in R(i, \varphi, G) : uv \in E\}|. \\
\lambda(uv, \varphi, G) &= |\varphi(u) - \varphi(v)|, \qquad uv \in E.
\end{aligned}
$$

Table 2: Layout problems and costs for a graph $G = (V, E)$ with $|V| = n$.

Problem	Name	Cost
Bandwidth	Bandwidth	$\mathrm{bw}(\varphi, G) = \max_{uv \in E} \lambda(uv, \varphi, G).$
Min. Lin. Arrangement	MinLA	$\mathrm{la}(\varphi, G) = \begin{cases} \sum_{uv \in E} \lambda(uv, \varphi, G), \\ \sum_{i=1}^{n} \theta(i, \varphi, G). \end{cases}$
Cutwidth	Cutwidth	$\mathrm{cw}(\varphi, G) = \max_{i=1}^{n} \theta(i, \varphi, G).$
Modified Cut	ModCut	$\mathrm{mc}(\varphi, G) = \sum_{i=1}^{n} \zeta(i, \varphi, G).$
Vertex Separation	VertSep	$\mathrm{vs}(\varphi, G) = \max_{i=1}^{n} \delta(i, \varphi, G).$
Pathwidth	Pathwidth	$\mathrm{pw}(\varphi, G) = \mathrm{vs}(\varphi, G).$
Sum Cut	SumCut	$\mathrm{sc}(\varphi, G) = \sum_{i=1}^{n} \delta(i, \varphi, G).$
Profile	Profile	$\mathrm{pr}(\varphi, G) = \begin{cases} \sum_{u \in V}(\varphi(u) - \min_{v \in \Gamma^*(u)} \varphi(v)), \\ \mathrm{sc}(\varphi^R, G). \end{cases}$
Edge Bisection	EdgeBis	$\mathrm{eb}(\varphi, G) = \theta(\lfloor n/2 \rfloor, \varphi, G).$
Vertex Bisection	VertBis	$\mathrm{vb}(\varphi, G) = \delta(\lfloor n/2 \rfloor, \varphi, G).$

(a) $G = (V, E)$.

(b) Graphical representation of φ. Dividing the layout at $i = 4$, left vertices are shown in light yellow, right vertices in dark green, cut edges and separator vertices are bold.
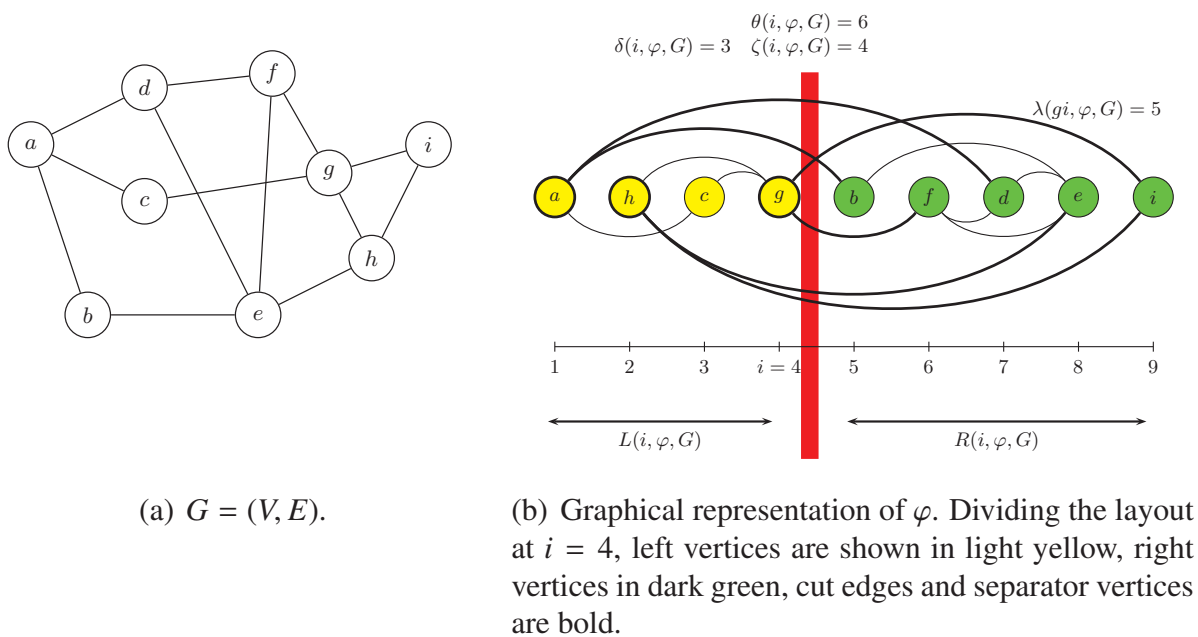
Figure 1: A graph G together with some layout measures and a graphical representation of the layout $\varphi = \{(a, 1), (b, 5), (c, 3), (d, 7), (e, 8), (f, 6), (g, 4), (j, 9), (h, 2)\}$.

Graphical representation. A common way to represent a layout φ of a graph G is to align its vertices on a horizontal line, mapping each vertex u to position $\varphi(u)$, as shown in Figure 1. This graphical representation gives an easy understanding of the previously defined measures: By drawing a vertical line just after position i and before position $i + 1$, the vertices at the left of the line belong to $L(i, \varphi, G)$ and the vertices at the right of the line belong to $R(i, \varphi, G)$. It is easy to compute the cut $\theta(i, \varphi, G)$ by counting the number of edges that cross the vertical line. The modified cut $\zeta(i, \varphi, G)$ counts all the edges in $\theta(i, \varphi, G)$ except those that have vertex $\varphi^{-1}(i)$ as endpoint. It is also easy to compute the separation $\delta(i, \varphi, G)$ by counting the number of vertices at the left of the vertical line that are joined with some vertex at the right of the vertical line. Finally, the length $\lambda(uv, \varphi, G)$ of an edge uv corresponds to the natural distance between its endpoint images.

Layout costs. A *layout cost* is a function F that associates to each layout φ of a graph G an integer $F(\varphi, G)$. Let F be a layout cost; the optimization layout problem associated with F consists in determining some layout $\varphi^* \in \Phi(G)$ of an input graph G such that

$$F(\varphi^*, G) = \min_{\varphi \in \Phi(G)} F(\varphi, G).$$

For any F and G, we define $\text{MIN}F(G) = \min_{\varphi \in \Phi(G)} F(\varphi, G)$.

Layout problems. The particular costs we consider are listed below, together with the layout problems they give raise to:

- *Bandwidth* (BANDWIDTH): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{BW}(\varphi^*, G) = \text{MINBW}(G)$ where

$$\text{BW}(\varphi, G) = \max_{uv \in E} \lambda(uv, \varphi, G).$$

- *Minimum Linear Arrangement* (MINLA): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{LA}(\varphi^*, G) = \text{MINLA}(G)$ where

$$\text{LA}(\varphi, G) = \sum_{uv \in E} \lambda(uv, \varphi, G) = \sum_{i=1}^{n} \theta(i, \varphi, G).$$

- *Cutwidth* (CUTWIDTH): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{cw}(\varphi^*, G) = \text{MINCW}(G)$ where

$$\text{cw}(\varphi, G) = \max_{i \in [|V|]} \theta(i, \varphi, G).$$

- *Modified Cut* (MODCUT): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{MC}(\varphi^*, G) = \text{MINMC}(G)$ where

$$\text{MC}(\varphi, G) = \sum_{i \in [|V|]} \zeta(i, \varphi, G).$$

- *Vertex Separation* or *Pathwidth* (VERTSEP/PATHWIDTH): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{vs}(\varphi^*, G) = \text{MINVS}(G)$ where

$$\text{vs}(\varphi, G) = \max_{i \in [|V|]} \delta(i, \varphi, G).$$

- *Sum Cut* (SUMCUT): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{sc}(\varphi^*, G) = \text{MINSC}(G)$ where

$$\text{sc}(\varphi, G) = \sum_{i \in [|V|]} \delta(i, \varphi, G).$$

- *Profile* (PROFILE): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{PR}(\varphi^*, G) = \text{MINPR}(G)$ where

$$\text{PR}(\varphi, G) = \sum_{u \in V} \left(\varphi(u) - \min_{v \in \Gamma^*(u)} \varphi(v) \right) \text{ and } \Gamma^*(u) = \{u\} \cup \{v \in V \mid uv \in E\}.$$

The PROFILE and SUMCUT problems are equivalent.

- *Edge Bisection* (EDGEBIS): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{EB}(\varphi^*, G) = \text{MINEB}(G)$ where

$$\text{EB}(\varphi, G) = \theta(\left\lfloor \tfrac{1}{2}|V| \right\rfloor, \varphi, G).$$

- *Vertex Bisection* (VERTBIS): Given a graph $G = (V, E)$, find a layout $\varphi^* \in \Phi(G)$ such that $\text{VB}(\varphi^*, G) = \text{MINVB}(G)$ where

$$\text{VB}(\varphi, G) = \delta(\left\lfloor \tfrac{1}{2}|V| \right\rfloor, \varphi, G).$$

Although the Edge Bisection and Vertex Bisection problems are not strictly layout problems, they fit well in our framework for layout problems.

The definitions of the previous problems are summarized in Table 2.

3   Results

The results of addenda are mainly organized as entries in the following tables. In order to distinguish the new results, their respective entries are marked with a little star ($\star$) in the tables. Because some results have been published first in conference proceedings and then in journals, table entries are sorted topologically, rather than strictly chronologically.

Table 3: Complexity results

BW	in general	[Papadimitriou, 1976]
	for trees with maximum degree 3	[Garey et al., 1978]
	for caterpillars with hair-length ≤ 3	[Monien, 1986]
	for caterpillars with ≤ 1 hair per backbone vertex	[Monien, 1986]
	for cyclic caterpillars	[Muradyan, 1999]
	★ for cyclic caterpillars with hair length 1	[Muradian, 2003]
	for grid graphs and unit disk graphs	[Díaz et al., 2001]
LA	in general	[Garey et al., 1976]
	for bipartite graphs	[Even et al., 1975]
	★ for interval graphs	[Cohen et al., 2006]
	★ for permutation graphs	[Cohen et al., 2006]
CW	in general	[Gavril, 1977]
	for graphs with maximum degree 3	[Makedon et al., 1985]
	for planar graphs with maximum degree 3	[Monien et al., 1988]
	for grid graphs and unit disk graphs	[Díaz et al., 2001]
	★ for split graphs and chordal graphs	[Heggernes et al., 2008]
MC	for planar graphs with maximum degree 3	[Monien et al., 1988]
PW	in general	[Lengauer, 1981]
	for planar graphs with maximum degree 3	[Monien et al., 1988]
	for ★ starlike and chordal graphs	[Gustedt, 1993]
	for bipartite graphs	[Goldberg et al., 1995]
	for grid graphs and unit disk graphs	[Díaz et al., 2001]
	★ for split graphs	[Gustedt, 1993]
SC	in general	[Díaz et al., 1991]
		[Lin et al., 1994b]
		[Golovach, 1997]
	for cobipartite graphs	[Yuan et al., 1998]
	★ for split graphs	[Peng et al., 2006]
EB	in general	[Garey et al., 1976]
	for graphs with maximum degree 3	[MacGregor, 1978]
	for graphs with bounded maximum degree	[MacGregor, 1978]
	for d-regular graphs	[Bui et al., 1987]

Table 4: Graphs optimally solvable in polynomial time I

BW	caterpillars with hair-length ≤ 2	$O(n \log n)$	[Assman et al., 1981]
	hypercubes	$O(n \log n)$	[Harper, 1966]
	$\star$ hypercubes		[Wang et al., 2009]
	butterflies	$O(n \log n)$	[Lai, 1997]
	interval graphs	$O(n\Delta^2 \log \Delta)$	[Muradyan, 1986]
	interval graphs	$O(n \log n)$	[Mahesh et al., 1991]
	interval graphs	$O(n \log n)$	[Sprague, 1994]
	chain graphs	$O(n^2 \log n)$	[Kloks et al., 1998]
	complete k-level t-ary tree	$O(n)$	[Heckmann et al., 1998]
	$\star$ rectangular grids	$O(1)$	[FitzGerald, 1974]
	$\star$ cubic grids	$O(1)$	[FitzGerald, 1974]
	square grids	$O(n)$	[Mai et al., 1984]
	$\star$ toroidal meshes	$O(1)$	[Paterson et al., 1993]
	$\star$ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
	$\star$ 3-dimensional grids	$O(1)$	*[Otachi et al., 2011]
LA	trees	$O(n^3)$	[Goldberg et al., 1976]
	rooted trees	$O(n \log n)$	[Adolphson et al., 1973]
	trees	$O(n^{2.2})$	[Shiloach, 1979]
	trees	$O(n^{\log 3/ \log 2})$	[Chung, 1988]
	rectangular grids	$O(n)$	[Muradyan et al., 1980]
	square grids	$O(n)$	[Mitchison et al., 1986]
	2-dimensional cylinder	$O(n)$	[Muradyan, 1982]
	hypercubes	$O(n)$	[Harper, 1964]
	de Bruijn graph of order 4	$O(n)$	[Harper, 1970]
	d-dimensional c-ary cliques	$O(n)$	[Ellis et al., 1964]
	complete p-partite graphs	$O(n + p \log p)$	[Muradyan et al., 1988]
	$\star$ proper interval graphs	$O(n)$	[Safro, 2002]
	$\star$ certain Halin graphs	$O(n)$	[Easton et al., 1996]
	$\star$ Outerplanar graphs	$O(\delta^2 n + n^2)$	[Frederickson et al., 1988]
	$\star$ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
	$\star$ chord graphs	$O(n \log n)$	[Rostami et al., 2008]
SC	trees	$O(n^{2.3})$	[Lepin, 1986]
	trees	$O(n)$	[Díaz et al., 1991]
	trees	$O(n^{1.722})$	[Kuo et al., 1994]
	interval graphs	?	[Lin et al., 1994a]
	$\star$ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
	$\star$ d-trapezoid and permutation graphs	$O(n^d)$	[Bodlaender et al., 1998]
	square grids	$O(n)$	[Díaz et al., 2000]
	$\star$ cographs	$O(n)$	[Fomin et al., 2000]
	$\star$ primitive starlike graphs	pol(n)	[Peng et al., 2006]
	$\star$ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]

Table 5: Graphs optimally solvable in polynomial time II

cw	trees	$O(n \log^{\Delta-2} n)$	[Chung et al., 1982]
	trees	$O(n \log n)$	[Yannakakis, 1985]
	hypercubes	$O(n)$	[Harper, 1964]
	d-dimensional c-ary cliques	$O(n)$	[Nakano, 1994]
	max degree $\leq \Delta$ and treewidth $\leq k$	$O(n^{\Delta k^2})$	[Thilikos et al., 2001]
	ordinary 2- and 3-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	toroidal and cylindrical 2-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	toroidal 3-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	complete p-partite graphs	$O(n + p \log p)$	[Muradyan et al., 1988]
	⋆ complete binary trees	$O(n)$	[Lengauer, 1982]
	⋆ abelian Cayley graphs	$O(1)$	[Berend et al., 2008]
	⋆ de Bruijn graphs	?	[Raspaud et al., 1995]
	⋆ bounded degree partial w-tree	$n^{O(wd)}$	[Thilikos et al., 2005b]
	⋆ threshols graphs	$O(n)$	[Heggernes et al., 2008]
	⋆ bipartite permutation graphs	$O(n)$	[Heggernes et al., 2010]
	⋆ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
pw	trees	$O(n \log n)$	[Ellis et al., 1979]
	trees	$O(n)$	[Skodinis, 2000]
	cographs	$O(n)$	[Bodlaender et al., 1993]
	permutation graphs	$O(n^2)$	[Bodlaender et al., 1995b]
	n-dimensional grids	$O(n^2)$	[Bollobás et al., 1991]
	⋆ bounded degree partial w-tree	$n^{O(wd)}$	[Thilikos et al., 2005b]
	⋆ unicyclic graphs	$O(n)$	[Ellis et al., 2004]
	⋆ 2-dimensional grids, cylinders, tori	$O(n)$	[Ellis et al., 2008]
	⋆ 3-dimensional grids	$O(1)$	[Otachi et al., 2011]*
	⋆ k-starlike, split, primitive starlike graphs		[Peng et al., 2000]
	⋆ trapezoid and permutation graphs	$O(n^2)$	[Bodlaender et al., 1998]
	⋆ biconvex bipartite graphs	$O(n)$	[Peng et al., 2007]
	⋆ circular-arc graphs	$O(n)$	[Suchan et al., 2007]
	⋆ cographs	$O(n)$	[Bodlaender et al., 1993]
	⋆ comparability graphs of interval orders	$O(n)$	[Garbe, 1995]

Table 6: Graphs optimally solvable in polynomial time III

MC	★ unit interval graphs	$O(n)$	[Jinjiang et al., 1995]
EB	trees	$O(n^3)$	[MacGregor, 1978]
	hypercubes	$O(n)$	[Nakano, 1994]
	d-dimensional c-ary arrays	$O(n)$	[Nakano, 1994]
	d-dimensional c-ary cliques	$O(n)$	[Nakano, 1994]
	ordinary 2- and 3-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	toroidal and cylindrical 2-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	toroidal 3-dim. meshes	$O(n^2)$	[Rolim et al., 1995]
	grid graphs	$O(n^5)$	[Papadimitriou et al., 1996]
	treewidth $\leq k$	$O(n^2)$	[Soumyanath et al., 1990]
	cube-connected cycles graphs	$O(n)$	[Manabe et al., 1984]
	★ Abelian Cayley graphs	?	[Berend et al., 2008]

Table 7: Fixed parameterized complexity results

Bandwidth(2)	$O(n)$	[Garey et al., 1978]
Bandwidth(k)	$O(n^{k+1})$	[Saxe, 1980]
Bandwidth(k)	$O(n^k)$	[Gurari et al., 1984]
Bandwidth(k)	$\mathbf{W}[k]$	[Bodlaender et al., 1994]
Cutwidth(2)	$O(n)$	[Garey et al., 1978]
Cutwidth(k)	$O(n^k)$	[Gurari et al., 1984]
Cutwidth(k)	$O(n^{k-1})$	[Makedon et al., 1989]
Cutwidth(k)	$O(n^2)$	[Fellows et al., 1992]
★ Cutwidth(k)	$O(n)$	[Thilikos et al., 2005a]
ModCut(k)	$O(n^2)$	[Fellows et al., 1992]
★SumCut(k)	FPT	[Gutin et al., 2006b]
Pathwidth(k)	$O(n^2)$	[Fellows et al., 1988]
Pathwidth(k)	$O(n)$	[Bodlaender, 1996]
MinLA($n + k$)	$O(m + n + 5.88^k)$	[Gutin et al., 2006a]

Table 8: Approximability results I

BW	3-appr for δ-dense graphs	$n^{O(1/\delta)}$	[Karpinski et al., 1997]
	2-appr for AT-free graphs	$O(n^3)$	[Kloks et al., 1999]
	$O(\log n)$-appr for caterpillars	$O(n^2)$	[Haralambides et al., 1991]
	$O(\log n)$-appr for GHB-trees	$O(n^2)$	[Haralambides et al., 1997]
	rnd $O(\log^{4.5} n)$-appr	$O(m(\lg^4 n \lg\lg n)$	[Feige, 2000]
	rnd $O(\log^3 n \sqrt{\log\log n})$-appr	pol(n)	[Dunagan et al., 2001]
	rnd $O(\sqrt{n/\text{BW}(G)}\, \log n)$-appr	pol(n)	[Blum et al., 2000]
	rnd $O(\log^{2.5} n)$-approx. for trees and for chordal graphs	pol(n)	[Gupta, 2001]
	no **PTAS**	—	[Blache et al., 1998]
	no **PTAS** for trees	—	[Blache et al., 1998]
	no **APX**	—	[Unger, 1998]
	★ no **APX** for caterpillars	—	[Dubey et al., 2011]
PW	$O(\log^2 n)$-appr	pol(n)	[Bodlaender et al., 1995a]
	$O(\log n)$-appr for planar graphs	pol(n)	[Bodlaender et al., 1995a]
	★ $O(\log^{1.5} n)$-appr	?	[Feige et al., 2005]
	★ 3-appr for outerplanar graphs	$O(n)$	[Govindan et al., 1998]
	★ 3-appr for Halin graphs	$O(n)$	[Fomin et al., 2006]
	★ not appr within an additive constant	—	[Bodlaender et al., 1995a]

Table 9: Approximability results II

|			
LA	**PTAS** for dense graphs	$n^{O(1/\epsilon^2)}$	[Arora et al., 1996]
	$O(\log^2 n)$-appr	pol(n)	[Hansen, 1989]
	$O(\log^2 n)$-appr	pol(n)	[Leighton et al., 1999]
	$O(\log n \log \log n)$-appr	pol(n)	[Even et al., 2000]
	$O(\log n)$-appr	pol(n)	[Rao et al., 1998]
	★ $O(\sqrt{\log n} \log \log n)$-appr	pol(n)	[Charikar et al., 2006]
			[Feige et al., 2007]
	$O(\log \log n)$-appr for planar graphs	pol(n)	[Rao et al., 1998]
	★ 4-appr for interval graphs	pol(n)	[Safro, 2002]
	★ 2-appr for interval graphs	pol(n)	[Cohen et al., 2006]
	★ no **APX** under Unique Games Conjecture	—	[Devanur et al., 2006]
	★ no **APX** if **NP-C** is not in rnd. sub-exp time	—	[Ambuhl et al., 2007]
CW	**PTAS** for dense graphs	$n^{O(1/\epsilon^2)}$	[Arora et al., 1996]
	$O(\log^2 n)$-appr	pol(n)	[Leighton et al., 1999]
SC	★ $O(\log^2 n)$-appr	pol(n)	[Ravi et al., 1991]
	$O(\log n \log \log n)$-appr	pol(n)	[Even et al., 2000]
	★ $O(\log n)$-appr for planar graphs	pol(n)	[Even et al., 2000]
	$O(\log n)$-appr	pol(n)	[Rao et al., 1998]
	$O(\log \log n)$-appr for planar graphs	pol(n)	[Rao et al., 1998]
EB	**PTAS** for dense graphs	$f(n, \epsilon)$	[Frieze et al., 1996]
	$O(\log^2 n)$-appr	pol(n)	[Feige et al., 2006]
	$O(\log^{1.5} n)$-appr	pol(n)	[Kao, 2008]
	$O(\log n)$-appr for planar graph	pol(n)	[Feige et al., 2006]
	★ no **APX** if **NP-C** not in rnd. sub-exp time	—	[Khot, 2006]
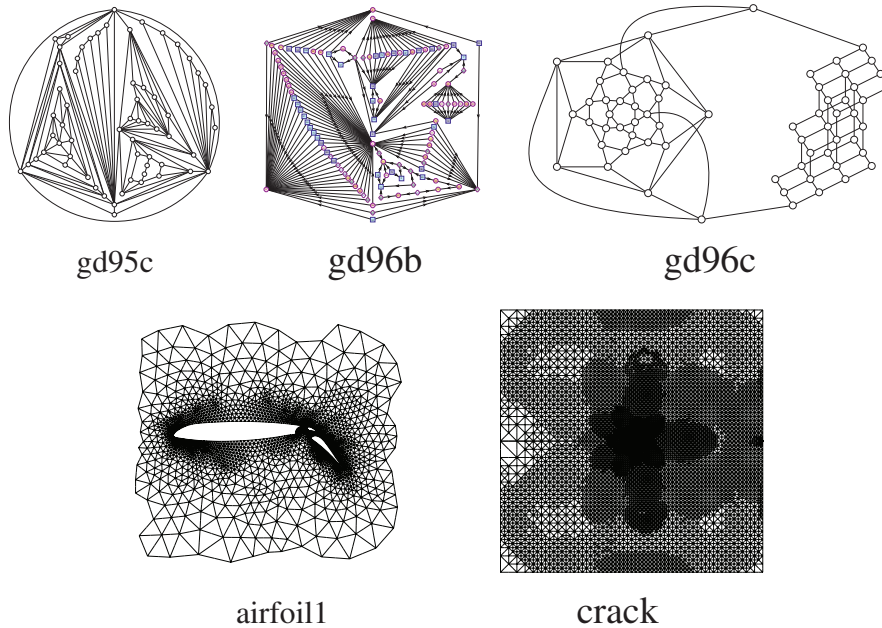
Figure 2: Sample graphs from the MINLA test suite

4 Practical results for MINLA

After having reviewed theoretical results on several layout problems, we turn now to practical results for the Minimum Linear Arrangement problem. We first discuss heuristic methods to get upper bounds, then we consider lower bounds.

Upper bounds. A first comparison of various heuristics for the MINLA problem was reported in [Petit, 1998, Petit, 2003]. This work stablished a test suite of graphs that has become an standard benchmark for this problem. Figure 2 illustrates some of these graphs. The algorithms considered included Successive Augmentation heuristics, Local Search heuristics and Spectral Sequencing. [Petit, 1998] concluded that the best approximations are usually obtained using Simulated Annealing, which involves a large amount of computation time wheras solutions found with Spectral Sequencing are close to the ones found with Simulated Annealing and can be obtained in significantly less time. Taken these considerations into account, [Petit, 2000] combined Spectral Sequencing and Simulated Annealing in a parallel setting to improve both time and solution quality.

The work of [Koren et al., 2002] presented a novel algorithm based on the multi-scale paradigm. Its results are on a par to those of Simulated Annealing, but the running time is significantly faster. Consequently it can be applied to larger graphs (ten thousands of vertices).

In [Poranen, 2005], a Genetic Hillclimbing algorithm is presented and analized on the same test suite. The solutions delivered by Genetic Hillclimbing improve those delivered by Simulated Annealing in quality and runtime in some

Heuristic	Reference	airfoil1	gd95c	gd96c	c1y
SS+SA	[Petit, 2003]	288977	506	519	63858
Genetic	[Poranen, 2005]	306005	506	519	64610
Multi-scale	[Koren et al., 2002]	291794	509	519	64934
Memetic	[Rodriguez-Tello et al., 2006]	285429	506	519	62333
Multi-level	[Safro et al., 2006]	272931	506	519	66262
2stage-SA	[Rodriguez-Tello et al., 2008]	276381	506	519	62230

Table 10: MɪɴLA upper bounds computed by several heuristics on different graphs.

graphs, but not in others. Similarly, in [Rodriguez-Tello et al., 2006], a Memetic algorithm was presented. This algorithm uses a greedy algorithm to generate an initial population, a fine tuned Simulated Annealing algorithm, and a specialized crossover operator. The solutions obtained with this Memetic algorithm are better than those obtained with the Genetic Hillclimbing.

A major improvement on heuristics for MɪɴLA was [Safro et al., 2006], which presented a Multi-level algorithm with weighted edge contraction. This method generates better results in less time and, again, even larger graphs can be considered.

The Two-Stage Simulated Annealing in [Rodriguez-Tello et al., 2008] integrates an heuristic to generate good quality initial solutions, a discriminating evaluation function, a special neighborhood space and an effective cooling schedule. This heuristic can be considered the current MɪɴLA champion, improving several of the previous best results.

Table 10 tries to summarize the above results for a minimalistic benchmark made of four graphs.

Lower bounds. A comparison of various simple lower bounds was also provided in [Petit, 2003]. These included the Degree and Edge methods (both based on the local connectivity of each vertex), the Gomory-Hu tree method (based on fundamental cuts), and the Mesh method (aimed to get lower bounds for graphs arising from finite element methods). The results made evident the existence of a a big gap between the best obtained upper bounds and the best obtained lower bounds.

In [Caprara et al., 2011], so called Decorous lower bounds were presented. These are based on a linear-programming approach, with variables representing distances and an exponential number of constraints. For the smallest grahs in the test suite, the lower bounds stablish gaps of 5–20% with respect to the state-of-the-art heuristics.

A further improvement was obtained in [Schwarz, 2010], which presents a novel approach based on a branch-and-cut algorithm using so called betweenness variables. Coupled with the upper bounds found by various heuristics, this method stablishes the opti-

Graph	best UB	[Petit, 2003]	[Caprara et al., 2011]	[Schwarz, 2010]
c1y	62230	14101	59971	
c2y	78757	17842	76253	
gd95c	**506**	292	443	**506**
gd96a	95263	5155	76253	
gd96b	**1416**	702	1281	**1416**
gd96c	**519**	241	402	**519**
gd96d	2393	595	2021	2391

Table 11: MᴵɴLA lower bounds computed by several methods on different graphs.

mality of some linear arrangements for the smallest graphs in the test suite (the have less than 120 vertices) and shows the high quality of the heuristics in medium sized graphs (arround 1000 vertices).

Table 11 gives a feeling on the lower bounds obtained with the above techniques and how they compare with the best upper bounds.

5   Conclusions

This paper extended the survey in [Díaz et al., 2002] with new results that have appeared in the literature up to the present.

Some of the new theoretical results show how the thin line that defines the hardness of some layout problems is being drawn more and more precisely. This can clearly be observed in the case of the Bᴀɴᴅᴡɪᴅᴛʜ problem: researchers have identified many particular classes of graphs where this problem remains **NP**-complete, or can be efficiently solved or cannot be approximated. Likewise, many similar results have been obtained in the latest ten years on the VᴇʀᴛSᴇᴘ problem, mainly because of its equivalence with the Pᴀᴛʜᴡɪᴅᴛʜ problem, which has received a great interest due to its relation with the graph minors theory.

With regard to the experimental results for the MᴵɴLA problem, one can remark that the current research on lower bounding techniques based on linear programming is very promising, having stablished in some cases that heuristics yield solutions close to the optima. This may suggest that further development of heuristics for this problem is not worth, wheras the current lower bounding methods should be improved to be applied to larger graphs.

Regarding the open problems stated in the concluding remarks of the 2002 original survey, I could not find any reference tackling them.

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